

Reduction formula of form factors for the integrable spin- s XXZ chains and application to the correlation functions

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Abstract

For the integrable spin- s XXZ chain we express explicitly any given spin- s form factor in terms of a sum over the scalar products of the spin-1/2 operators. Here they are given by the operator-valued matrix elements of the monodromy matrix of the spin-1/2 XXZ spin chain. In the paper we call an arbitrary matrix element of a local operator between two Bethe eigenstates a form factor of the operator. We derive all important formulas of the fusion method in detail. We thus revise the derivation of the higher-spin XXZ form factors given in a previous paper. The revised method has several interesting applications in mathematical physics. For instance, we express the spin- s XXZ correlation function of an arbitrary entry at zero temperature in terms of a sum of multiple integrals.

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1 Introduction

The multiple-integral representations of correlation functions of the spin-1/2 XXZ spin chain have attracted much interest during the last two decades in the mathematical physics of integrable quantum spin chains [1, 2, 3, 4, 5, 6, 7, 8]. They are also derived for the integrable higher-spin XXX spin chains through the algebraic Bethe-ansatz method [9, 10]. The multiple-integral representations of the finite-temperature correlation functions of the integrable isotropic spin-1 chain have been explicitly derived [11].

The Hamiltonian of the spin-1/2 XXZ spin chain under the periodic boundary conditions (P.B.C.) is given by

$$\mathcal{H}_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^L (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z) . \quad (1.1)$$

Here σ_j^a ($a = X, Y, Z$) are the Pauli matrices defined on the j th site and Δ denotes the anisotropy of the exchange coupling. The P.B.C. are given by $\sigma_{L+1}^a = \sigma_1^a$ for $a = X, Y, Z$. In terms of the q parameter of the quantum group $U_q(sl_2)$, we express Δ by $\Delta = \frac{1}{2}(q + q^{-1})$. We define parameter η by $q = \exp \eta$. The transfer matrix of the XXZ spin chain has free parameters which we call the inhomogeneity parameters w_j for $j = 1, 2, \dots, L$.

Recently, a systematic method for evaluating the form factors and correlation functions of the integrable higher-spin XXZ spin chain has been proposed by applying the fusion method [12, 13, 14]. However, the proposed method was not completely correct [15]. In the paper, we revise the previous method for evaluating the spin- $\ell/2$ form factors, and formulate systematic formulas by which we can express any given spin- $\ell/2$ form factor in terms of a sum over the scalar products of the spin-1/2 operators. We show the derivation of the revised method, explicitly. As an application of the fusion method we derive a concise multiple-integral representation for the spin- s XXZ correlation function of an arbitrary entry in a region of the gapless regime, which is now expressed in terms of a sum of the multiple integrals. Here we remark that we call an arbitrary matrix element of a local spin- s operator between two Bethe eigenvectors a form factor of the operator in the paper, following Refs. [2, 6].

Let us review the fusion method for evaluating the higher-spin form factors briefly, and point out where it was wrong. We consider the integrable spin- $\ell/2$ XXZ spin chain for an integer ℓ with $\ell > 1$. In the fusion method we construct the spin- $\ell/2$ XXZ transfer matrix in the following two steps: we first construct the spin-1/2 transfer matrix with $w_j = w_j^{(\ell)}$ from the product of the spin-1/2 R -matrices with their rapidities shifted by the inhomogeneity parameters w_j which are given by the N_s pieces of the complete ℓ -strings $w_j^{(\ell)}$ such as $w_j = w_j^{(\ell)}$ for $j = 1, 2, \dots, L$ (see §2.4); we then multiply the product with the spin- $\ell/2$ projection operators. Here, the spin- $\ell/2$ chain with N_s sites is defined on the spin-1/2 chain with L sites, where $L = \ell N_s$.

When we evaluate the spin- $\ell/2$ form factors with the fusion method, we reduce each of the spin- $\ell/2$ operators into a sum of products of the local spin-1/2 operators, which we want to express in terms of the operator-valued matrix elements of the spin-1/2 monodromy matrix

through the formula of the quantum inverse-scattering problem (QISP). We then want to calculate the expectation value or the matrix elements of the sums of products of the local spin-1/2 operators with respect to the Bethe states by making use of Slavnov's formula of scalar products for the spin-1/2 Bethe-ansatz operators. Here we recall that the inhomogeneity parameters w_j of the spin-1/2 monodromy matrix are given by the N_s pieces of the complete ℓ -strings, $w_j = w_j^{(\ell)}$, for $j = 1, 2, \dots, L$.

However, the QISP formula does not hold, if one of the transfer matrices appearing in it is nonregular. Here we remark that it has the product of the inverse operators of the transfer matrices where the spectral parameters λ are given by some of inhomogeneity parameters w_j . In fact, we can show that the spin-1/2 transfer matrix with $w_j = w_j^{(\ell)}$ is non-regular at $\lambda = w_{\ell(k-1)+1}^{(\ell)}$, the first rapidity of the k th complete ℓ -string for an integer k of $1 \leq k \leq N_s$ (see §3.6). Consequently, the QISP formula does not hold in the straightforward form for the fusion method. Thus, we want to avoid such special values of the spectral parameter when we evaluate the matrix elements or expectation values of the higher-spin local operators through the fusion method. In the revised method we avoid directly putting the complete ℓ -strings $w_j^{(\ell)}$ into the inhomogeneity parameters w_j , as we shall see later.

The main result of the present paper should have several interesting applications in the mathematical physics of exactly solvable models. For instance, with the revised method we can evaluate the form factors of various solvable quantum spin chains associated with the affine quantum group [16]. Then, for the spin- s XXZ spin chain we can derive the multiple-integral representation of correlation functions through the revised method, as we have mentioned in the above. Moreover, it should be an interesting problem to calculate some form factors and matrix elements of the local spin operators for the superintegrable chiral Potts chains through the present method [17, 18]. Furthermore, there is another interesting possible application. We can construct integrable quantum impurity models such as consisting of one spin- S site with N spin-1/2 sites through the fusion method. We can then calculate the form factors for the spin- S site and the correlation functions between the spin- S site and the other spin-1/2 sites, by applying the method in the present paper. These interesting topics should be discussed in separate papers.

The paper consists of the following. In section 2 we introduce the finite-dimensional representations of the quantum group and construct the monodromy matrix of the integrable higher-spin XXZ spin chain through the fusion method. We then introduce the complete ℓ -strings, $w_j^{(\ell)}$, where the k th complete ℓ -string $w_{\ell(k-1)+\alpha}^{(\ell)}$ for $1 \leq \alpha \leq \ell$ is given by the sequence of ℓ complex numbers shifted by η successively, such as $\xi_k, \xi_k - \eta, \dots, \xi_k - (\ell - 1)\eta$. In section 3, we define the higher-spin elementary operators $E^{i,j}$, which have only one nonzero matrix element of entry (i, j) with respect to the basis vectors and their conjugate vectors. Then, we explicitly derive a formula (Proposition 3.7), by which we can reduce any given product of the higher-spin elementary operators into a sum of products of the spin-1/2 elementary operators. In order to prove it we derive two expressions for a given product of the spin-1/2 elementary

operators. Interestingly, the overall factor for the form factor of a higher-spin operator depends on whether the operator is associated with principal grading or homogeneous grading. In order to make the form factors independent of the grading w we define the general spin- $\ell/2$ elementary operators $\widehat{E}^{i,j(\ell w)}$. We show in §3.6 that the spin-1/2 transfer matrix with $w_j = w_j^{(\ell)}$ is non-regular at $\lambda = w_{\ell(k-1)+1}^{(\ell)} = \xi_k$. In section 4, we reduce the form factor of a product of the spin- $\ell/2$ elementary operators into those of the spin-1/2 elementary operators (Proposition 4.2). We introduce the *almost complete ℓ -strings*, $w_j^{(\ell;\epsilon)}$ ($1 \leq j \leq L$), a set of inhomogeneity parameters that are slightly different from the complete ℓ -strings $w_j^{(\ell)}$ by the order of a small parameter ϵ . We reduce the spin- $\ell/2$ form factors into a sum of the spin-1/2 scalar products by making use of the QISP formula with inhomogeneity parameters given by the almost complete ℓ -strings and by sending ϵ to 0 [19]. We can revise the expressions of the higher-spin form factors given in Ref. [12] making use of Proposition 4.5 [20]. In section 5, we show an explicit formula by which we can calculate every spin- $\ell/2$ form factor in terms of the scalar products of the spin-1/2 operators. Finally, in section 6, we express the correlation function of an arbitrary entry for the integrable spin- s XXZ spin chain in terms of a sum of the multiple integrals. Moreover, the normalization factors are systematically shown for the general spin- $\ell/2$ elementary operators $\widehat{E}^{i,j(\ell w)}$. Here, the expression of the correlation functions is different from that of Ref. [13] mainly with respect to the sum over the multiple integrals [21]. Due to the spin inversion symmetry, however, the spin-1 one point functions are expressed in terms of single multiple integrals, which are the same with those of Ref. [13], .

2 Affine quantum group and the monodromy matrix

2.1 Spin- $\ell/2$ representations of the quantum group $U_q(sl_2)$

The quantum algebra $U_q(sl_2)$ is an associative algebra over \mathbf{C} generated by X^\pm, K^\pm with the following relations: [22, 23, 24]

$$\begin{aligned} KX^\pm K^{-1} &= q^{\pm 2}X^\pm, \quad KK^{-1} = K^{-1}K = 1, \\ [X^+, X^-] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned} \quad (2.1)$$

The algebra $U_q(sl_2)$ is also a Hopf algebra over \mathbf{C} with comultiplication

$$\begin{aligned} \Delta(X^+) &= X^+ \otimes 1 + K \otimes X^+, \quad \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-, \\ \Delta(K) &= K \otimes K, \end{aligned} \quad (2.2)$$

and antipode: $S(K) = K^{-1}$, $S(X^+) = -K^{-1}X^+$, $S(X^-) = -X^-K$, and coproduct: $\epsilon(X^\pm) = 0$ and $\epsilon(K) = 1$.

Let us introduce some symbols of q -analogues. For any given integer n we define the q -integer of n by $[n]_q = (q^n - q^{-n})/(q - q^{-1})$, and the q -factorial of n by

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q. \quad (2.3)$$

For integers m and n satisfying $m \geq n \geq 0$ we define the q -binomial coefficients as follows

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q! [n]_q!}. \quad (2.4)$$

We first formulate the spin-1/2 representation $V^{(1)}$. Let $|\alpha\rangle$ ($\alpha = 0, 1$) be the basis vectors. We have $X^-|0\rangle = |1\rangle$, $X^-|1\rangle = 0$, $X^+|0\rangle = 0$, $X^+|1\rangle = |0\rangle$, and $K|\alpha\rangle = q^{1-2\alpha}|\alpha\rangle$ for $\alpha = 0, 1$.

We now construct the spin- $\ell/2$ representation $V^{(\ell)}$ of $U_q(sl_2)$ for a non-negative integer ℓ , and in the tensor product space $(V^{(1)})^{\otimes \ell}$ of the spin-1/2 representations $V^{(1)}$. Let us introduce the basis vectors of $V^{(\ell)}$, $||\ell, n\rangle$, for $n = 0, 1, \dots, \ell$. First, we define the highest weight vector $||\ell, 0\rangle$ by

$$||\ell, 0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \dots \otimes |0\rangle_\ell. \quad (2.5)$$

Here $|\alpha\rangle_j$ for $\alpha = 0, 1$ denote the basis vectors of the spin-1/2 representation defined on the j th component of the tensor product $(V^{(1)})^{\otimes \ell}$. We remark that 0 and 1 corresponds to up-spin, \uparrow , and down-spin, \downarrow , respectively. We also remark that $K||\ell, 0\rangle = q^\ell ||\ell, 0\rangle$. We define $||\ell, n\rangle$ for $n \geq 1$ by

$$||\ell, n\rangle = (\Delta^{(\ell-1)}(X^-))^n ||\ell, 0\rangle \frac{1}{[n]_q!}. \quad (2.6)$$

We then have the following: [12]

$$||\ell, i\rangle = \sum_{1 \leq a(1) < \dots < a(i) \leq \ell} \sigma_{a(1)}^- \dots \sigma_{a(i)}^- ||\ell, 0\rangle q^{a(1)+a(2)+\dots+a(i)-i\ell+i(i-1)/2} \quad \text{for } i = 0, 1, \dots, \ell. \quad (2.7)$$

Here the sum is taken over all such integers $a(1), a(2), \dots, a(i)$ that satisfy $1 \leq a(1) < \dots < a(i) \leq \ell$. We denote by σ_j^\pm the Pauli matrices acting on the j th site.

We denote by $F(\ell, n)$ the “square length”s of vectors $||\ell, n\rangle$ as follows.

$$F(\ell, n) = (||\ell, n\rangle)^T \cdot ||\ell, n\rangle. \quad (2.8)$$

Here the superscript T denotes the matrix transposition. Setting $(||\ell, 0\rangle)^T \cdot ||\ell, 0\rangle = 1$, we have

$$F(\ell, n) = \begin{bmatrix} \ell \\ n \end{bmatrix}_q q^{-n(\ell-n)}. \quad (2.9)$$

We thus define conjugate vectors $\langle \ell, j|$ by

$$\langle \ell, j| = (||\ell, j\rangle)^T / F(\ell, j) \quad \text{for } j = 0, 1, \dots, \ell. \quad (2.10)$$

Explicitly we have the following:

$$\langle \ell, j| = \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} q^{j(\ell-j)} \sum_{1 \leq b(1) < \dots < b(j) \leq \ell} \langle \ell, 0| \sigma_{b(1)}^+ \dots \sigma_{b(j)}^+ q^{b(1)+b(2)+\dots+b(j)-j\ell+j(j-1)/2}. \quad (2.11)$$

It is easy to show the normalization factor (2.9) by making use of the following lemma:

Lemma 2.1. For an integer n with $0 < n \leq \ell$ we have

$$\sum_{1 \leq a(1) < \dots < a(n) \leq \ell} q^{2a(1) + \dots + 2a(n)} = q^{n(\ell+1)} \begin{bmatrix} \ell \\ n \end{bmatrix}_q. \quad (2.12)$$

Proof. Let us consider the q -binomial theorem:

$$\prod_{k=1}^{\ell} (1 - zq^{2k}) = \sum_{n=0}^{\ell} (-1)^n z^n q^{n(n+1)} \begin{bmatrix} \ell \\ n \end{bmatrix}_q. \quad (2.13)$$

Expanding the left hand side of (2.13) with respect to z we have

$$\prod_{k=1}^{\ell} (1 - zq^{2k}) = \sum_{n=0}^{\ell} (-z)^n \sum_{1 \leq a(1) < \dots < a(n) \leq \ell} q^{2a(1) + \dots + 2a(n)}. \quad (2.14)$$

Hence we have (2.12). \square

We remark that when q is complex and not real, the conjugate vector $\langle \ell, j |$ is different from the Hermitian conjugate of a given vector $||\ell, j\rangle$. Thus, the pairing of $\langle \ell, j |$ and $||\ell, k\rangle$ does not give a standard scalar product if q is complex and not real. However, the Hermitian conjugate of a vector $||\ell, j\rangle$ is not covariant with respect to the quantum group $U_q(sl_2)$ if q is complex and not real, while the transposed vector is covariant. Therefore, we express the transposed vector as the conjugate vector $\langle \ell, j |$.

We now introduce the projection operator which maps the tensor product of the spin-1/2 representations $(V^{(1)})^{\otimes \ell}$ to the spin- ℓ representation $V^{(\ell)}$. In terms of the basis vectors and their conjugate vectors we define it by

$$P^{(\ell)} = \sum_{n=0}^{\ell} ||\ell, n\rangle \langle \ell, n|. \quad (2.15)$$

We shall denote it also by $P_{1 \dots \ell}^{(\ell)}$, since it is defined on the tensor product $(V^{(1)})^{\otimes \ell}$.

2.2 Operators in the tensor product space

Let us consider the tensor product $V_1^{(\ell)} \otimes \dots \otimes V_{N_s}^{(\ell)}$ of the $(\ell + 1)$ -dimensional vector spaces $V_j^{(\ell)}$ with parameter λ_j for $j = 1, 2, \dots, N_s$. Here we assume $L = \ell N_s$. We call the tensor product $V_1^{(\ell)} \otimes \dots \otimes V_{N_s}^{(\ell)}$ the quantum space. In the most general cases, we consider the tensor product of the auxiliary space $V_0^{(2s_0)}$ and the quantum space $(V_1^{(2s_1)} \otimes \dots \otimes V_r^{(2s_r)})$ such as $V_0^{(2s_0)} \otimes (V_1^{(2s_1)} \otimes \dots \otimes V_r^{(2s_r)})$ with $2s_1 + \dots + 2s_r = L$, where $V_j^{(2s_j)}$ have spectral parameters λ_j for $j = 1, 2, \dots, r$. Here s_j are given by integers or half-integers for $j = 0, 1, 2, \dots, r$.

We denote by $e^{a,b}$ such two-by-two matrices that have only one nonzero element equal to 1 at the entry of (a, b) for $a, b = 0, 1$. We also express by $E^{a,b(2s)}$ the $(2s+1)$ -by- $(2s+1)$ matrices

with unique nonzero element 1 at the entry of (a, b) for $a, b = 0, 1, \dots, 2s$. If it is defined on the j th component of the quantum space, we denote it by $E_j^{a, b(2s)}$. For a given set of matrix elements $\mathcal{A}_{b, \beta}^{a, \alpha}$ for $a, b = 0, 1, \dots, 2s_j$ and $\alpha, \beta = 0, 1, \dots, 2s_k$, we define operators $\mathcal{A}_{j, k}$ by

$$\mathcal{A}_{j, k} = \sum_{a, b=1}^{2s_j} \sum_{\alpha, \beta=1}^{2s_k} \mathcal{A}_{b, \beta}^{a, \alpha} I^{(2s_0)} \otimes I^{(2s_1)} \otimes \dots \otimes I^{(2s_{j-1})} \otimes E^{a, b(2s_j)} \otimes I^{(2s_{j+1})} \otimes \dots \otimes I^{(2s_{k-1})} \otimes E^{\alpha, \beta(2s_k)} \otimes I^{(2s_{k+1})} \otimes \dots \otimes I^{(2s_r)}. \quad (2.16)$$

Similarly, for a set of matrix elements \mathcal{B}_b^a for $a, b = 0, 1, \dots, 2s_j$, we define operators \mathcal{B}_j by

$$\mathcal{B}_j = \sum_{a, b=1}^{2s_j} \mathcal{B}_b^a I^{(2s_0)} \otimes I^{(2s_1)} \otimes \dots \otimes I^{(2s_{j-1})} \otimes E^{a, b(2s_j)} \otimes I^{(2s_{j+1})} \otimes \dots \otimes I^{(2s_r)}. \quad (2.17)$$

2.3 R -matrix and the monodromy matrix of types $(1, 1^{\otimes L})$

Let us introduce the R -matrix of the XXZ spin chain [1, 5, 6, 7]. Let V_1 and V_2 be two-dimensional vector spaces. The R -matrix acting on $V_1 \otimes V_2$ associated with homogeneous grading of type $w = +$ is given by

$$R_{12}^{(1+)}(\lambda_1 - \lambda_2) = \sum_{a, b, c, d=0,1} R^{(1+)}(u)_{cd}^{ab} e_1^{a, c} \otimes e_2^{b, d} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^-(u) & 0 \\ 0 & c^+(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]}, \quad (2.18)$$

where $u = \lambda_1 - \lambda_2$, $b(u) = \sinh u / \sinh(u + \eta)$ and $c^\pm(u) = \exp(\pm u) \sinh \eta / \sinh(u + \eta)$. Here, the suffix [1, 2] in eq. (2.18) denotes that the matrix acts on the tensor product of V_1 and V_2 .

We remark that the R -matrix associated with homogeneous grading of type $w = -$, $R_{12}^{(1-)}(\lambda_1 - \lambda_2)$, is given by exchanging all the \pm signs in (2.18) [12, 13].

In the massless regime, we set $\eta = i\zeta$ by a real number ζ , and we have $\Delta = \cos \zeta$. In the paper we mainly consider the region $0 \leq \zeta < \pi/2s$ for the correlation functions. In the massive regime, we assign η a real nonzero number and we have $\Delta = \cosh \eta > 1$.

We denote by $R^{(1p)}(u)$ or simply by $R(u)$ the symmetric R -matrix where $c^\pm(u)$ of (2.18) are replaced by $c(u) = \sinh \eta / \sinh(u + \eta)$ [12]. The symmetric R -matrix is compatible with principal grading of the affine quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ [12].

Let us now consider the $(L + 1)$ th tensor product of the spin-1/2 representations, which consists of the tensor product of the auxiliary space $V_0^{(1)}$ and the quantum space which is given by the L th tensor product of $V_j^{(1)}$ for $j = 1, 2, \dots, L$; i.e., $V_0^{(1)} \otimes (V_1^{(1)} \otimes \dots \otimes V_L^{(1)})$. We call the tensor product that of type $(1, 1^{\otimes L})$.

Let $\{w_j\}_L$ denote the set of free parameters w_j for $j = 1, 2, \dots, L$. We call them inhomogeneity parameters. We define the spin-1/2 XXZ monodromy matrices associated with grading of types $w = \pm, p$ by

$$T_{0, 12 \dots L}^{(1, 1^w)}(\lambda; \{w_j\}_L) = R_{0, 12 \dots L}^{(1^w)} = R_{0L}^{(1^w)}(\lambda - w_L) \cdots R_{01}^{(1^w)}(\lambda - w_1). \quad (2.19)$$

Here $R_{jk}^{(1w)}$ denote the R -matrices associated with grading of types $w = \pm$ and p with inhomogeneity parameters $\{w_j\}_L$.

2.4 Rapidities forming n -strings

We introduce a set of rapidities for a positive integer n , which we call a complete n -string. It is given by a set of n rapidities of the following form:

$$\lambda_j = \Lambda + (n - 2j + 1)\eta/2, \quad \text{for } j = 1, 2, \dots, n. \quad (2.20)$$

Here we call parameter Λ the center of the complete n -string.

Let ϵ be an infinitesimally small number; i.e., we have $|\epsilon| \ll 1$. We take generic parameters r_b for $b = 1, 2, \dots, n$. We define an “almost complete n -string” by the following set of n rapidities

$$\lambda_j = \Lambda + (n - 2j + 1)\eta/2 + \epsilon r_b, \quad \text{for } j = 1, 2, \dots, n. \quad (2.21)$$

They are different from the complete n -string with center Λ by the small numbers ϵr_b .

We introduce N_s sets of almost complete ℓ -strings $w_j^{(\ell; \epsilon)}$ for $1 \leq j \leq L$ as follows.

$$w_j^{(\ell; \epsilon)} = \xi_b - (\beta - 1)\eta + \epsilon r_b^{(\beta)} \quad \text{for } \beta = 1, 2, \dots, \ell; b = 1, 2, \dots, N_s. \quad (2.22)$$

Here we assume that parameters $r_b^{(\beta)}$ are generic. We recall $\ell N_s = L$.

When $\epsilon = 0$ the set of inhomogeneity parameters $w_j^{(\ell; \epsilon)}$ gives N_s pieces of complete ℓ -strings. We denote them by $w_j^{(\ell)}$; i.e., $w_j^{(\ell)} = w_j^{(\ell; 0)}$ for $j = 1, 2, \dots, L$. We shall put $w_j^{(\ell; \epsilon)}$ and $w_j^{(\ell)}$ into the homogeneous parameters w_j ($1 \leq j \leq L$) of the spin-1/2 transfer matrix defined on the L sites, later.

2.5 Monodromy matrix of type $(1, \ell^{\otimes N_s})$

We shall now construct the spin- $\ell/2$ monodromy matrix on the spin- $\ell/2$ chain with N_s sites. We define L by $L = \ell N_s$. We consider the tensor product of the spin-1/2 auxiliary space $V^{(1)}$ and the N_s th tensor product $(V^{(\ell)})^{\otimes N_s}$ of quantum spaces $V^{(\ell)}$; i.e. $V^{(1)} \otimes (V^{(\ell)})^{\otimes N_s}$. Here we construct $(V^{(\ell)})^{\otimes N_s}$ in $(V^{(1)})^{\otimes L}$.

In the fusion construction of the higher-spin monodromy matrices it is useful to introduce the following symbols for the spin-1/2 monodromy matrices:

$$T_{0, 12 \dots L}^{(1, \ell w; \epsilon)}(\lambda) = T^{(1, 1w)}(\lambda; \{w_j^{(\ell; \epsilon)}\}_L), \quad \text{for } w = \pm, p, \quad (2.23)$$

where we put $w_j = w_j^{(\ell; \epsilon)}$ for $j = 1, 2, \dots, L$. It is nothing but the spin-1/2 monodromy matrix $T^{(1, 1w)}(\lambda)$ with special inhomogeneity parameters.

We introduce the following symbol:

$$T^{(1, \ell+; 0)}(\lambda) = \lim_{\epsilon \rightarrow 0} T^{(1, \ell+; \epsilon)}(\lambda). \quad (2.24)$$

We thus express the $(0,1)$ -element of the spin-1/2 monodromy matrix $T^{(1,\ell+;0)}(\lambda)$ as

$$B^{(\ell+;0)}(\lambda) = (T^{(1,\ell+;0)}(\lambda))_{0,1}. \quad (2.25)$$

Let us denote by $P_{\ell(b-1)+1}^{(\ell)}$ the projection operator which maps the tensor product of the spin-1/2 representations $V_{\ell(b-1)+1}^{(1)} \otimes \cdots \otimes V_{\ell(b-1)+\ell}^{(1)}$ to the b th component of the N_s th tensor product $(V^{(\ell)})^{\otimes N_s}$. Here b is an integer satisfying $1 \leq b \leq N_s$, and the tensor product $V_{\ell(b-1)+1}^{(1)} \otimes \cdots \otimes V_{\ell(b-1)+\ell}^{(1)}$ corresponds to the $\ell(b-1)+1$ th to $\ell(b-1)+\ell$ th components of the L th tensor product $(V^{(1)})^{\otimes L}$. We define $P_{1\dots L}^{(\ell)}$ by

$$P_{1\dots L}^{(\ell)} = \prod_{b=1}^{N_s} P_{\ell(b-1)+1}^{(\ell)}. \quad (2.26)$$

We construct the spin- $\ell/2$ monodromy matrix $T_{0,12\dots N_s}^{(1,\ell+)}(\lambda)$ associated with homogeneous grading by applying the projection operator $P_{1\dots L}^{(\ell)} = \prod_{b=1}^{N_s} P_{\ell(b-1)+1}^{(\ell)}$ as follows [12].

$$T_{0,12\dots N_s}^{(1,\ell+)}(\lambda; \{\xi_b\}_{N_s}) = P_{1\dots L}^{(\ell)} T_{0,12\dots L}^{(1,\ell+;0)}(\lambda) P_{1\dots L}^{(\ell)}. \quad (2.27)$$

2.6 Higher-spin monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$

We set the inhomogeneity parameters w_j for $j = 1, 2, \dots, L$, as N_s sets of complete $2s$ -strings [12]. We define $w_{(b-1)\ell+\beta}^{(2s)}$ for $\beta = 1, \dots, 2s$, as follows.

$$w_{2s(b-1)+\beta}^{(2s)} = \xi_b - (\beta - 1)\eta, \quad \text{for } b = 1, 2, \dots, N_s. \quad (2.28)$$

We now define the monodromy matrix of type $(1, (2s)^{\otimes N_s})$ associated with homogeneous grading. We first recall the definition of the monodromy matrix as follows.

$$\begin{aligned} T_{0,12\dots N_s}^{(1,2s+)}(\lambda_0; \{\xi_b\}_{N_s}) &= P_{12\dots L}^{(2s)} R_{0,1\dots L}^{(1,1+)}(\lambda_0; \{w_j^{(2s)}\}_L) P_{12\dots L}^{(2s)} \\ &= \begin{pmatrix} A^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) & B^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \\ C^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) & D^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \end{pmatrix}. \end{aligned} \quad (2.29)$$

We shall now define the monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ associated with homogeneous grading. It acts on the tensor product of the auxiliary space $V_{a_1\dots a_\ell}$ and the quantum space $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$. Let us express the tensor product $V_0^{(\ell)} \otimes (V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)})$ by the following symbol

$$(\ell, (2s)^{\otimes N_s}) = (\ell, \overbrace{2s, 2s, \dots, 2s}^{N_s}). \quad (2.30)$$

Here we recall that $V_0^{(\ell)}$ abbreviates $V_{a_1 a_2 \dots a_\ell}^{(\ell)}$. For the auxiliary space $V_0^{(\ell)}$ we define the monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$T_{0,12\dots N_s}^{(\ell,2s+)} = P_{a_1 a_2 \dots a_\ell}^{(\ell)} T_{a_1,12\dots N_s}^{(1,2s+)}(\lambda_{a_1}) T_{a_2,12\dots N_s}^{(1,2s+)}(\lambda_{a_1} - \eta) \cdots T_{a_\ell,12\dots N_s}^{(1,2s+)}(\lambda_{a_1} - (\ell - 1)\eta) P_{a_1 a_2 \dots a_\ell}^{(2s)}. \quad (2.31)$$

Here we remark that it is associated with homogeneous grading.

2.7 Gauge transformations

Let us consider a two-by-two diagonal matrix $\Phi(w) = \text{diag}(1, \exp(w))$. In terms of the tensor-product notation (2.17) we introduce $\Phi_j(w)$ for $j = 0, 1, \dots, L$, in the tensor product $(V^{(1)})^{\otimes L}$ of the spin-1/2 representations $V^{(1)}$. We define the gauge transformation $\chi_{12\dots L}$ by

$$\chi_{12\dots L} = \Phi_1(w_1)\Phi_2(w_2)\cdots\Phi_L(w_L). \quad (2.32)$$

Here w_j denote the inhomogeneity parameters of the spin-1/2 transfer matrix of the XXZ spin chain for $j = 1, 2, \dots, L$.

In the N_s th tensor product of the spin- $\ell/2$ representations, $(V^{(\ell)})^{\otimes N_s}$, which we call the quantum space of the spin- $\ell/2$ XXZ spin chain, we now introduce the gauge transformation $\chi_{12\dots N_s}^{(\ell)}$ for the spin- $\ell/2$ monodromy matrix $T^{(1, \ell+)}(\lambda)$ with inhomogeneity parameters $w_{\ell(k-1)}^{(\ell)}$ for $k = 1, 2, \dots, N_s$. Here we recall that they are given by the complete ℓ -strings (2.22) with parameters ξ_b for $b = 1, 2, \dots, N_s$. We introduce the $(\ell+1)$ -by- $(\ell+1)$ diagonal matrix $\Phi^{(\ell)}(w)$ by

$$\Phi^{(\ell)}(w) ||\ell, n\rangle = \exp(nw) ||\ell, n\rangle \quad \text{for } n = 0, 1, \dots, \ell. \quad (2.33)$$

We then define the spin- $\ell/2$ gauge transformation $\chi_{12\dots N_s}^{(\ell)}$ on the quantum space $(V^{(\ell)})^{\otimes N_s}$ by

$$\chi_{12\dots N_s}^{(\ell)} = \Phi_1^{(\ell)}(\Lambda_1)\Phi_2^{(\ell)}(\Lambda_2)\cdots\Phi_{N_s}^{(\ell)}(\Lambda_{N_s}), \quad (2.34)$$

where Λ_b denote the string centers as follows:

$$\Lambda_b = \xi_b - (\ell-1)\eta/2, \quad \text{for } b = 1, 2, \dots, N_s. \quad (2.35)$$

Considering the tensor product of the auxiliary space $V^{(1)}$ and the quantum space $(V^{(\ell)})^{\otimes N_s}$ we define the gauge transformation $\chi_{0,12\dots N_s}^{(1, \ell)}$ on $V^{(\ell)} \otimes (V^{(\ell)})^{\otimes N_s}$ by

$$\chi_{0,12\dots N_s}^{(1, \ell)} = \Phi_0\Phi_1^{(\ell)}\cdots\Phi_{N_s}^{(\ell)}. \quad (2.36)$$

Similarly, we define $\chi_{0,12\dots L}$ by $\chi_{0,12\dots L} = \Phi_0(\lambda)\Phi_1(w_1)\cdots\Phi_L(w_L)$.

2.8 Spin- $\ell/2$ monodromy matrices associated with principal grading

We now construct the higher-spin monodromy matrix of type $(1, (\ell)^{\otimes N_s})$ associated with principal grading. We denote it by $T_{0,12\dots N_s}^{(1, \ell p)}(\lambda)$, which acts on the quantum space $V_1^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)}$. In the fusion construction we define it by

$$\begin{aligned} T_{0,12\dots N_s}^{(1, \ell p)}(\lambda) &= \left(\chi_{0,12\dots N_s}^{(1, \ell)}\right)^{-1} T_{0,12\dots N_s}^{(1, \ell+)}(\lambda) \left(\chi_{0,12\dots N_s}^{(1, \ell)}\right) \\ &= \left(\chi_{0,12\dots N_s}^{(1, \ell)}\right)^{-1} \left(P_{12\dots L}^{(\ell)} T_{0,12\dots L}^{(1, \ell+; 0)}(\lambda) P_{12\dots L}^{(\ell)}\right) \chi_{0,12\dots N_s}^{(1, \ell)}. \end{aligned} \quad (2.37)$$

Let us construct the higher-spin monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ associated with principal grading, which acts on the quantum space $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$. From the higher-spin

monodromy matrices associated with homogeneous grading we derive them through the inverse of the gauge transformation as follows

$$T^{(\ell, 2s p)} = \left(\chi_{a_1 \dots a_\ell, 12 \dots N_s}^{(\ell, 2s)} \right)^{-1} T^{(\ell, 2s+)}(\lambda) \left(\chi_{a_1 \dots a_\ell, 12 \dots N_s}^{(\ell, 2s)} \right). \quad (2.38)$$

Here $\chi_{a_1 \dots a_\ell, 12 \dots N_s}^{(\ell, 2s)}$ denote the following:

$$\chi_{a_1 \dots a_\ell, 12 \dots N_s}^{(\ell, 2s)} = \Phi_{a_1 \dots a_\ell}^{(\ell)}(\Lambda_0) \Phi_1^{(2s)}(\Lambda_1) \dots \Phi_{N_s}^{(2s)}(\Lambda_{N_s}), \quad (2.39)$$

where Λ_0 denotes the string center, $\Lambda_0 = \lambda_{a_1} - (\ell - 1)\eta/2$.

Hereafter we shall denote $T^{(1, \ell, w)}(\lambda)$ by $T^{(\ell, w)}(\lambda)$, briefly.

3 Reduction of higher-spin elementary operators

3.1 Spin- $\ell/2$ elementary operators associated with homogeneous and principal gradings

Let us consider the spin- $\ell/2$ representation $V^{(\ell)}$ constructed in the ℓ th tensor product space $(V^{(1)})^{\otimes \ell}$. We define the spin- $\ell/2$ elementary operators associated with homogeneous grading, $E^{i, j(\ell+)}$, by

$$E^{i, j(\ell+)} = ||\ell, i\rangle\langle\ell, j|| \quad \text{for } i, j = 0, 1, \dots, \ell. \quad (3.1)$$

We define the spin- $\ell/2$ elementary matrices associated with principal grading, $E^{i, j(\ell p)}$, also by

$$E^{i, j(\ell p)} = ||\ell, i\rangle\langle\ell, j|| \quad \text{for } i, j = 0, 1, \dots, \ell. \quad (3.2)$$

In the paper we define it by the same operator as that of homogeneous grading. We have

$$E^{i, j(\ell+)} = E^{i, j(\ell p)} = ||\ell, i\rangle\langle\ell, j||. \quad (3.3)$$

Through (2.15), which expresses the projection operator in terms of the basis vectors and conjugate vectors, we have the following:

Lemma 3.1. *In the $(\ell + 1)$ -dimensional representation $V^{(\ell)}$ for the spin- $\ell/2$ elementary operators with grading of $w = \pm, p$ and the spin- $\ell/2$ projection operator we have*

$$P^{(\ell)} E^{i, j(\ell w)} = E^{i, j(\ell w)} P^{(\ell)} = E^{i, j(\ell w)}. \quad (3.4)$$

Let us recall that we have set $L = \ell N_s$, and the quantum space $(V^{(\ell)})^{\otimes N_s}$ is constructed in the L th tensor product space $(V^{(1)})^{\otimes L}$. We now introduce the spin- $\ell/2$ elementary operators associated with grading of w , $E_k^{i, j(\ell w)}$, acting on the k th component of the quantum space $(V^{(\ell)})^{\otimes N_s}$ as follows.

$$E_k^{i, j(\ell w)} = (I^{(\ell)})^{\otimes (k-1)} \otimes E^{i, j(\ell w)} \otimes (I^{(\ell)})^{\otimes (N_s - k)} \quad \text{for } k = 1, 2, \dots, N_s. \quad (3.5)$$

3.2 Two expressions of a product of spin-1/2 elementary operators

For a given product of the spin-1/2 elementary operators we shall express it in another form. Let us first consider the simplest example. In terms of the highest weight vector $||\ell, 0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_\ell$ we have the following:

$$\begin{aligned} ||\ell, 0\rangle\langle\ell, 0|| &= |0\rangle_1 \otimes \cdots \otimes |0\rangle_\ell \langle 0|_1 \otimes \cdots \otimes \langle 0|_\ell \\ &= |0\rangle_1 \langle 0|_1 \otimes \cdots \otimes |0\rangle_\ell \langle 0|_\ell \\ &= e_1^{0,0} \cdots e_\ell^{0,0}. \end{aligned} \quad (3.6)$$

Thus, the product of the spin-1/2 elementary operators $e_1^{0,0} \cdots e_\ell^{0,0}$ is also expressed as $||\ell, 0\rangle\langle\ell, 0||$. Here we remark that $\langle 0|_1 = (1, 0)$ and $|0\rangle_1 = (1, 0)^T$, where the superscript T denotes the matrix transposition. We thus have

$$|0\rangle_1 \langle 0|_1 = (1, 0)^T (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e^{0,0}. \quad (3.7)$$

We shall generalize relation (3.6) in the following.

Let us introduce symbols for expressing sequences. If a sequence of numbers, a_1, a_2, \dots, a_N , are given, we denote it by $(a_j)_N$, briefly; i.e., we have

$$(a_j)_N = (a_1, a_2, \dots, a_N). \quad (3.8)$$

Here we recall that we denote by $\{\mu_k\}_N$ a set of N parameters μ_k ; i.e., $\mu_1, \mu_2, \dots, \mu_N$.

We now consider two sequences consisting of only two values 0 or 1, $(\varepsilon'_\alpha)_\ell$ and $(\varepsilon_\beta)_\ell$. Here, the values of ε'_α and ε_β are given by 0 or 1 for $\alpha, \beta = 1, 2, \dots, \ell$. For given such sequences $(\varepsilon'_\alpha)_\ell$ and $(\varepsilon_\beta)_\ell$ we consider the following product of the spin-1/2 elementary operators:

$$\prod_{k=1}^{\ell} e_k^{\varepsilon'_k, \varepsilon_k} = e_1^{\varepsilon'_1, \varepsilon_1} \cdots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell}. \quad (3.9)$$

Here we recall that $e_k^{\varepsilon', \varepsilon}$ for $\varepsilon', \varepsilon = 0, 1$ denote the two-by-two matrices defined on the k th sites with unique nonzero element 1 at the entry $(\varepsilon', \varepsilon)$ for integers k satisfying $1 \leq k \leq \ell$.

Let us give another expression of product (3.9). We define a set $\boldsymbol{\alpha}^-$ by the set of integers k satisfying $\varepsilon'_k = 1$ for $1 \leq k \leq \ell$ and a set $\boldsymbol{\alpha}^+$ by the set of integers k satisfying $\varepsilon_k = 0$ for $1 \leq k \leq \ell$, respectively:

$$\boldsymbol{\alpha}^-(\{\varepsilon'_\alpha\}) = \{\alpha; \varepsilon'_\alpha = 1 (1 \leq \alpha \leq \ell)\}, \quad \boldsymbol{\alpha}^+(\{\varepsilon_\beta\}) = \{\beta; \varepsilon_\beta = 0 (1 \leq \beta \leq \ell)\}. \quad (3.10)$$

Let us denote by Σ_ℓ the set of integers $1, 2, \dots, \ell$; i.e., $\Sigma_\ell = \{1, 2, \dots, \ell\}$. In terms of sets $\boldsymbol{\alpha}^\pm$ we express the product of elementary operators given by (3.9) as

$$\prod_{a \in \boldsymbol{\alpha}^-} \sigma_a^- ||\ell, 0\rangle\langle\ell, 0|| \prod_{b \in \Sigma_\ell \setminus \boldsymbol{\alpha}^+} \sigma_b^+. \quad (3.11)$$

We now derive the expression of (3.9) from that of (3.11), in detail. Let us denote by r and r' the number of elements of the set α^- and α^+ , respectively. We express the elements of α^- as $a(k)$ for $k = 1, 2, \dots, r$, and those of $\Sigma_\ell \setminus \alpha^+$ as $b(k)$ for $k = 1, 2, \dots, r'$, respectively. Expressing r and $\ell - r'$ by i and j , respectively, we have

$$\alpha^- = \{a(1), a(2), \dots, a(i)\}, \quad \Sigma_\ell \setminus \alpha^+ = \{b(1), b(2), \dots, b(j)\}. \quad (3.12)$$

Hereafter if not specified, we shall put them in increasing order: $1 \leq a(1) < \dots < a(i) \leq \ell$ and $1 \leq b(1) < \dots < b(j) \leq \ell$, respectively. Here we recall $i = r$ and $j = \ell - r'$. We thus express the product of the elementary operators in terms of $a(k)$ and $b(k)$ as follows:

$$\begin{aligned} \prod_{a \in \alpha^-} \sigma_a^- ||\ell, 0\rangle \langle \ell, 0| \prod_{b \in \Sigma_\ell \setminus \alpha^+} \sigma_b^+ &= \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- ||\ell, 0\rangle \langle \ell, 0| \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ \\ &= e_{a(1)}^{1,0} \cdots e_{a(i)}^{1,0} e_1^{0,0} \cdots e_\ell^{0,0} e_{b(1)}^{0,1} \cdots e_{b(j)}^{0,1}. \end{aligned} \quad (3.13)$$

Calculating products of two-by-two matrices, from expression (3.11) we derive the expression in terms of products of the spin-1/2 elementary operators, $e_1^{\varepsilon'_1, \varepsilon_1} \cdots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell}$, such as given in (3.9). Here, we derive the sequence $(\varepsilon'_\alpha)_\ell$ by setting $\varepsilon'_{a(k)} = 1$ for $k = 1, 2, \dots, i$ while $\varepsilon'_\alpha = 0$ for $\alpha \neq a(k)$ with k of $1 \leq k \leq i$:

$$\varepsilon'_\alpha = \begin{cases} 1 & \text{if } \alpha = a(k) \ (1 \leq k \leq i), \\ 0 & \text{otherwise.} \end{cases} \quad (3.14)$$

Similarly, we derive sequence $(\varepsilon_\beta)_\ell$ by setting $\varepsilon_{b(k)} = 1$ for $k = 1, 2, \dots, j$ while $\varepsilon_\beta = 0$ for $\beta \neq b(k)$ with k of $1 \leq k \leq j$.

Let us introduce useful notation. Suppose that we have a sequence $(\varepsilon'_\alpha)_\ell$ such that $\varepsilon'_\alpha = 0$ or 1 for all integers α with $1 \leq \alpha \leq \ell$ and the number of integers α satisfying $\varepsilon'_\alpha = 1$ ($1 \leq \alpha \leq \ell$) is given by i . Then, we denote ε'_α by $\varepsilon'_\alpha(i)$ for each integer α and the sequence $(\varepsilon'_\alpha)_\ell$ by $(\varepsilon'_\alpha(i))_\ell$. In the same way, we denote by $(\varepsilon_\beta(j))_\ell$ a sequence of 0 or 1 such that the number of integers β satisfying $\varepsilon_\beta(j) = 1$ for $1 \leq \beta \leq \ell$ is given by j . The two expressions of a product of the spin-1/2 elementary operators are summarized as follows.

Lemma 3.2. *Sequences $(\varepsilon'_\alpha(i))_\ell$ and $(\varepsilon_\beta(j))_\ell$ are related to integers $a(1) < a(2) < \dots < a(i)$ and $b(1) < b(2) < \dots < b(j)$, respectively, by*

$$e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \cdots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} = e_{a(1)}^{1,0} \cdots e_{a(i)}^{1,0} e_1^{0,0} \cdots e_\ell^{0,0} e_{b(1)}^{0,1} \cdots e_{b(j)}^{0,1}, \quad (3.15)$$

$$\prod_{k=1}^i e_k^{\varepsilon'_k(i), \varepsilon_k(j)} = \prod_{a \in \alpha^-} \sigma_a^- ||\ell, 0\rangle \langle \ell, 0| \prod_{b \in \Sigma_\ell \setminus \alpha^+} \sigma_b^+. \quad (3.16)$$

3.3 Reduction into the spin-1/2 elementary operators

We shall express the spin- $\ell/2$ elementary operators $E^{i,j(\ell^+)}$ for integers i and j satisfying $1 \leq i, j \leq \ell$ in terms of sums of products of the spin-1/2 elementary matrices. It follows from

(2.7) and (2.11) that we have

$$||\ell, i\rangle\langle\ell, j|| = \sum_{(\varepsilon'_\alpha(i))_\ell} \sum_{(\varepsilon_\beta(j))_\ell} g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)}. \quad (3.17)$$

Here the sum is taken over all sequences $(\varepsilon'_\alpha(i))_\ell$ and $(\varepsilon_\beta(j))_\ell$. The coefficients $g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j))$ are given explicitly as follows.

$$g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) = \left[\begin{matrix} \ell \\ j \end{matrix} \right]_q^{-1} q^{(a(1)+\dots+a(i)+(b(1)+\dots+b(j))-(i+j)\ell+i(i-1)/2+j(j-1)/2}. \quad (3.18)$$

The ket vectors $\langle\ell, i||$ satisfy the following symmetry, which plays a central role in the fusion method for evaluating the spin- $\ell/2$ form factors.

Lemma 3.3. *Let α^- be a set of distinct integers $\{a(1), \dots, a(i)\}$ satisfying $1 \leq a(1) < \dots < a(i) \leq \ell$, we have the following:*

$$\langle\ell, i||\sigma_{a(1)}^- \dots \sigma_{a(i)}^- ||\ell, 0\rangle q^{-(a(1)+\dots+a(i))+i} = \left[\begin{matrix} \ell \\ i \end{matrix} \right]_q^{-1} q^{-i(i-1)/2}, \quad (3.19)$$

which is independent of the set $\alpha^- = \{a(1), a(2), \dots, a(i)\}$.

Proof. From the explicit expression (2.11) of the conjugate vector $\langle\ell, i||$ we have (3.19). \square

Expressing the matrix elements of the matrix $\Phi(w)$ as $(\Phi(w))_{a,b} = \delta(a,b) \exp(aw)$ for $a, b = 0, 1$, we show the gauge transformation $\chi_{12\dots\ell}$ on the spin-1/2 elementary operators as follows.

Lemma 3.4. *Recall that $\varepsilon'_\alpha(i)$ and $\varepsilon_\beta(j)$ are related to $a(k)$ and $b(k)$ via (3.15). Every product of the spin-1/2 elementary operators is transformed with the gauge transformation as*

$$\chi_{12\dots\ell} e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} \chi_{12\dots\ell}^{-1} = e_1^{\varepsilon'_1, \varepsilon_1} \dots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell} q^{-(a(1)+\dots+a(i)-i)+(b(1)+\dots+b(j)-j)} e^{(i-j)\xi_1}. \quad (3.20)$$

It is useful to express Lemma 3.3 in the following form.

Corollary 3.5. *For a pair of integers i and j with $1 \leq i, j \leq \ell$, let us consider sequences $(\varepsilon'_\alpha(i))_\ell$ and $(\varepsilon_\beta(j))_\ell$, which correspond to sets α^- and α^+ , respectively, through (3.15) and (3.16). The product of the spin-1/2 elementary operators multiplied by the projection operator from the left and multiplied also by $q^{-(a(1)+\dots+a(i))+i}$ does not depend on the set α^-*

$$P^{(\ell)} e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} q^{-(a(1)+\dots+a(i))+i} = \left[\begin{matrix} \ell \\ i \end{matrix} \right]_q^{-1} q^{-i(i-1)/2} ||\ell, i\rangle\langle\ell, 0|| \prod_{\beta \in \Sigma_\ell \setminus \alpha^+} \sigma_\beta^+. \quad (3.21)$$

In terms of the gauge transformation we express relation (3.21) as

$$P^{(\ell)} \chi_{12\dots\ell} e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} \chi_{12\dots\ell}^{-1} = \left[\begin{matrix} \ell \\ i \end{matrix} \right]_q^{-1} q^{-i(i-1)/2} ||\ell, i\rangle\langle\ell, 0|| \prod_{b \in \Sigma_\ell \setminus \alpha^+} \sigma_b^+ q^{b(1)+\dots+b(j)-j}. \quad (3.22)$$

Lemma 3.6. *The sum of coefficients $g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j))$ over all sequences $(\varepsilon'_\alpha(i))_\ell$ multiplied by $q^{a(1)+\dots+a(i)-i}$ is given by the following:*

$$\sum_{(\varepsilon'_\alpha(i))_\ell} g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) q^{a(1)+\dots+a(i)-i} = \left[\begin{matrix} \ell \\ i \end{matrix} \right]_q \left[\begin{matrix} \ell \\ j \end{matrix} \right]_q^{-1} q^{b(1)+\dots+b(j)-j} q^{i(i-1)/2-j(j-1)/2}. \quad (3.23)$$

Here we remark that we take the sum over all sequences of the form of $(\varepsilon'_\alpha(i))_\ell$.

Proof. Putting $n = i$ in (2.12) and observing that the sum over sequences $(\varepsilon'_\alpha(i))_\ell$ corresponds to the sum over integers $a(1), \dots, a(i)$ satisfying $1 \leq a(1) < \dots < a(i) \leq \ell$, we have (3.23) from (2.12). \square

We thus show the main formula for reducing the spin- $\ell/2$ operators into the spin-1/2 ones.

Proposition 3.7. *For every pair of integers i and j with $1 \leq i, j \leq \ell$ the spin- $\ell/2$ elementary operator associated with grading w , $E_1^{i,j(\ell w)}$, is decomposed into a sum of products of the spin-1/2 elementary operators as follows.*

$$E_1^{i,j(\ell w)} = \left[\begin{matrix} \ell \\ i \end{matrix} \right]_q \left[\begin{matrix} \ell \\ j \end{matrix} \right]_q^{-1} q^{i(i-1)/2-j(j-1)/2} e^{-(i-j)\xi_1} P_{12\dots\ell}^{(\ell)} \sum_{(\varepsilon_\beta(j))_\ell} \chi_{12\dots\ell} e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} \chi_{12\dots\ell}^{-1}. \quad (3.24)$$

Here, we fix a sequence $(\varepsilon'_\alpha(i))_\ell$. Furthermore, the expression (3.24) does not depend on the order of $\varepsilon'_\alpha(i)$ with respect to α s.

We shall show the derivation of Proposition 3.7 explicitly in Appendix A.

In terms of the string center: $\Lambda_1 = \xi_1 - (\ell - 1)\eta/2$, the q factors in eq. (3.24) are expressed as follows.

$$\begin{aligned} q^{i(i-1)/2-j(j-1)/2} e^{-(i-j)\xi_1} &= q^{-i(\ell-i)/2+j(\ell-j)/2} e^{-(i-j)(\xi_1-(\ell-1)\eta/2)} \\ &= q^{-i(\ell-i)/2+j(\ell-j)/2} e^{-(i-j)\Lambda_1}. \end{aligned} \quad (3.25)$$

Thus, introducing the symbol

$$N_{i,j}^{(\ell)} = \left[\begin{matrix} \ell \\ i \end{matrix} \right]_q \left[\begin{matrix} \ell \\ j \end{matrix} \right]_q^{-1} q^{-i(\ell-i)/2+j(\ell-j)/2}, \quad (3.26)$$

we express (3.24) compactly as follows

$$E_1^{i,j(\ell w)} = N_{i,j}^{(\ell)} e^{-(i-j)\Lambda_1} P_{12\dots\ell}^{(\ell)} \sum_{(\varepsilon_\beta(j))_\ell} \chi_{12\dots\ell} e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} \chi_{12\dots\ell}^{-1}. \quad (3.27)$$

In Ref. [13] the Hermitian elementary operators $\widetilde{E}^{i,j(\ell,+)}$ are introduced. The expectation values of the Hermitian elementary operators are the same as those of the standard elementary operators, $E^{i,j(\ell,+)}$. We shall show the reduction formula for the Hermitian elementary operators in Appendix B.

3.4 General spin- $\ell/2$ elementary operators

Let us consider a similarity transformation of the basis vectors as follows

$$||\ell, m\rangle \rightarrow ||\ell, m\rangle/g(m), \quad \langle \ell, n| \rightarrow g(n) \langle \ell, n|, \quad \text{for } m, n = 0, 1, \dots, \ell. \quad (3.28)$$

In the spin- $\ell/2$ representation constructed in the ℓ th tensor product space $(V^{(1)})^{\otimes \ell}$, we define the general spin- s elementary operators associated with principal grading, $\hat{E}^{i,j(\ell p)}$, by

$$\hat{E}^{i,j(\ell p)} = ||\ell, i\rangle \langle \ell, j| \frac{g(j)}{g(i)}, \quad \text{for } i, j = 0, 1, \dots, \ell. \quad (3.29)$$

Then, through the spin- $\ell/2$ gauge transformation we define the general spin- s elementary operators associated with homogeneous grading by

$$\hat{E}^{i,j(\ell+)} = \chi_{12\dots N_s}^{(\ell)} \hat{E}^{i,j(\ell p)} \left(\chi_{12\dots N_s}^{(\ell)} \right)^{-1}. \quad (3.30)$$

We explicitly have

$$\hat{E}^{i,j(\ell+)} = ||\ell, i\rangle \langle \ell, j| \frac{g(j)}{g(i)} e^{(i-j)(\xi - (\ell-1)\eta/2)}, \quad \text{for } i, j = 0, 1, \dots, \ell. \quad (3.31)$$

Here we recall that the quantity $\xi - (\ell-1)\eta/2$ corresponds to the string center of the ℓ -string: $\xi, \xi - \eta, \dots, \xi - (\ell-1)\eta$. They are originally the evaluation parameters of the ℓ th tensor product of the spin-1/2 representations, $(V^{(1)})^{\otimes \ell}$.

We remark that the definition of the general elementary operators $\hat{E}^{i,j(\ell w)}$ associated with grading w are covariant under the gauge transformations. We also remark that if we put $g(j) = \exp(j(\xi - (\ell-1)\eta/2))$, then expression (3.31) reduces to that of $E^{i,j(\ell+)}$.

We define the general spin- $\ell/2$ elementary operators associated with principal grading acting in the tensor product space $V_1^{(\ell)} \otimes \dots \otimes V_{N_s}^{(\ell)}$ by

$$\hat{E}_k^{i,j(\ell p)} = (I^{(\ell)})^{\otimes (k-1)} \otimes \hat{E}^{i,j(\ell p)} \otimes (I^{(\ell)})^{\otimes (N_s-k)}, \quad \text{for } i, j = 0, 1, \dots, \ell. \quad (3.32)$$

Similarly we define that of homogeneous grading, $\hat{E}_k^{i,j(\ell,+)}$ for $i, j = 0, 1, \dots, \ell$.

Let us introduce the normalization factor $\hat{N}_{i,j}^{(\ell)}$ by $\hat{N}_{i,j}^{(\ell)} = N_{i,j}^{(\ell)} g(i)/g(j)$. We have

$$\hat{N}_{i,j}^{(\ell)} = \frac{g(j)}{g(i)} \frac{F(\ell, i)}{F(\ell, j)} q^{i(\ell-i)/2 - j(\ell-j)/2}. \quad (3.33)$$

We define $\delta(w, p)$ for gradings \pm and p by

$$\delta(w, p) = \begin{cases} 1 & \text{if } w = p, \\ 0 & \text{otherwise.} \end{cases} \quad (3.34)$$

With factor $\hat{N}_{i,j}^{(\ell)}$ and the string center: $\Lambda_1 = \xi_1 - (\ell-1)\eta/2$, from Proposition 3.7, we have

$$\hat{E}_1^{i,j(\ell w)} = \hat{N}_{i,j}^{(\ell)} e^{-(i-j)\Lambda_1} \delta(w, p) P_{12\dots \ell}^{(\ell)} \sum_{(\varepsilon_\beta(j))_\ell} \chi_{12\dots \ell} e_1^{\varepsilon_1'(i), \varepsilon_1(j)} \dots e_\ell^{\varepsilon_\ell'(i), \varepsilon_\ell(j)} \chi_{12\dots \ell}^{-1}. \quad (3.35)$$

Here, we recall that sequence $(\varepsilon'_\alpha(i))_\ell$ is fixed. We also recall that Λ_1 denotes .

Let us recall $\Lambda_k = \xi_k - (\ell-1)\eta/2$ for $k = 1, 2, \dots, N_s$. We have the following.

Proposition 3.8. *In the k th component of the quantum space $(V^{(\ell)})^{\otimes N_s}$ for $1 \leq k \leq N_s$ we take integers i_k and j_k satisfying $0 \leq i_k, j_k \leq \ell$. The general spin- $\ell/2$ elementary operator, $E_k^{i_k j_k (\ell w)}$, is decomposed into a sum of products of the spin-1/2 elementary operators as follows*

$$\hat{E}_k^{i_k j_k (\ell w)} = \hat{N}_{i_k, j_k}^{(\ell)} e^{-(i-j)\Lambda_k \delta(w,p)} P_{\ell(k-1)+1}^{(\ell)} \sum_{(\epsilon_\beta(j_k))_\ell} \chi_{12\dots L} e_{\ell(k-1)+1}^{\epsilon'_1(i_k), \epsilon_1(j_k)} \dots e_{\ell(k-1)+\ell}^{\epsilon'_\ell(i_k), \epsilon_\ell(j_k)} \chi_{12\dots L}^{-1}. \quad (3.36)$$

Here we fix a sequence $(\epsilon'_\alpha(i))_\ell$. Furthermore, expression (3.36) does not depend on the order of $\epsilon'_\alpha(i)$ with respect to α .

Let us consider a product of the general spin- $\ell/2$ elementary operators, $\hat{E}_1^{i_1, j_1 (\ell w)} \dots \hat{E}_m^{i_m, j_m (\ell w)}$, for which we shall evaluate the zero-temperature spin- s XXZ correlation functions. We introduce variables $\epsilon_\alpha^{[k]'}(i_k)$ and $\epsilon_\beta^{[k]}(j_k)$ which take only two values 0 or 1 for $k = 1, 2, \dots, m$ and $\alpha, \beta = 0, 1, \dots, \ell$.

Corollary 3.9. *Let us take integers i_k and j_k satisfying $1 \leq i_k, j_k \leq \ell$ for $k = 1, 2, \dots, m$. The product of the general spin- $\ell/2$ elementary operators, $\hat{E}_1^{i_1, j_1 (\ell w)} \dots \hat{E}_m^{i_m, j_m (\ell w)}$, is expressed in terms of a sum of products of the spin-1/2 elementary operators as*

$$\begin{aligned} \prod_{k=1}^m \hat{E}_k^{i_k, j_k (\ell w)} &= \prod_{k=1}^m \left(\hat{N}_{i_k, j_k}^{(\ell)} e^{-(i_k - j_k)\Lambda_k \delta(w,p)} \right) \\ &\times P_{1\dots L}^{(\ell)} \sum_{(\epsilon_\beta^{[1]}(j_1))_\ell} \dots \sum_{(\epsilon_\beta^{[m]}(j_m))_\ell} \chi_{12\dots(\ell m)} \prod_{k=1}^m \left(e_{\ell(k-1)+1}^{\epsilon_1^{[k]'}(i_k), \epsilon_1^{[k]}(j_k)} \dots e_{\ell(k-1)+\ell}^{\epsilon_\ell^{[k]'}(i_k), \epsilon_\ell^{[k]}(j_k)} \right) \chi_{12\dots(\ell m)}^{-1}. \end{aligned} \quad (3.37)$$

Here we fix $(\epsilon_\alpha^{[k]'}(i_k))_\ell$ for each integer k of $1 \leq k \leq m$.

The expression (3.37) is useful for deriving the multiple-integral representations of correlation functions for the integrable higher-spin XXZ spin chain, as we shall see in §6.

3.5 Quantum inverse-scattering problem for the spin- $\ell/2$ operators

Let us recall the formula of the quantum inverse-scattering problem (QISP) for the spin-1/2 XXZ spin chain [6, 7].

$$x_n = \prod_{k=1}^{n-1} (A^{(1w)} + D^{(1w)})(w_k) \cdot \text{tr}_0 \left(x_0 T_{0,12\dots L}^{(1w)}(w_n) \right) \cdot \prod_{k=1}^n (A^{(1w)} + D^{(1w)})^{-1}(w_k). \quad (3.38)$$

Here we assume that inhomogeneity parameters w_j are given by generic values so that the transfer matrices $(A^{(1w)} + D^{(1w)})(w_k)$ are regular for $k = 1, 2, \dots, n$.

Making use of the QISP formula (3.38) we have the following expressions for $b = 1, 2, \dots, N_s$:

$$\begin{aligned} e_{\ell(b-1)+1}^{\epsilon'_1, \epsilon_1} \dots e_{\ell(b-1)+\ell}^{\epsilon'_\ell, \epsilon_\ell} &= \prod_{k=1}^{\ell(b-1)} (A^{(1w)}(w_k) + D^{(1w)}(w_k)) \\ &\times T_{\epsilon_1, \epsilon'_1}^{(1w)}(w_{\ell(b-1)+1}) \dots T_{\epsilon_\ell, \epsilon'_\ell}^{(1w)}(w_{\ell(b-1)+\ell}) \prod_{k=1}^{\ell b} (A^{(1w)}(w_k) + D^{(1w)}(w_k))^{-1}. \end{aligned} \quad (3.39)$$

Here we have denoted by $T_{\alpha,\beta}(\lambda)$ the (α, β) element of the monodromy matrix $T(\lambda)$.

Applying (3.39) to reduction formula (3.24) (or (3.36)) we obtain the QISP formula for the spin- $\ell/2$ local operators. For an illustration, we show the case of $b = 1$ as follows

$$\begin{aligned} \widehat{E}_1^{ij(\ell w)} &= \widehat{N}_{i,j}^{(\ell)} e^{-(i-j)\Lambda_1 \delta(w,p)} \times \\ &\times P_{1\dots\ell}^{(\ell)} \cdot \chi_{12\dots\ell} \sum_{(\varepsilon_\beta(j))_\ell} T_{\varepsilon_1(j), \varepsilon'_1(i)}^{(1w)}(w_1) \cdots T_{\varepsilon_\ell(j), \varepsilon'_\ell(i)}^{(1w)}(w_\ell) \prod_{k=1}^{\ell} (A^{(1w)}(w_k) + D^{(1w)}(w_k))^{-1} \chi_{12\dots\ell}^{-1}. \end{aligned} \quad (3.40)$$

Here, we fix a sequence $(\varepsilon'_\alpha(i))_\ell$.

3.6 Non-regularity of the transfer matrix at special points

Let us consider the sector of M down-spins on the spin-1/2 chain with L sites.

Proposition 3.10. *In the sector of M down-spins with $1 \leq M \leq L - 1$, the spin-1/2 transfer matrix $A^{(\ell w; 0)}(\lambda) + D^{(\ell w; 0)}(\lambda)$ is non-regular at $\lambda = w_{\ell(k-1)+1}^{(\ell)} + n\pi\sqrt{-1}$ for $k = 1, 2, \dots, N_s$ and $n \in \mathbf{Z}$. Here, $w_{\ell(k-1)+1}^{(\ell)} = \xi_k$ is the first rapidity of the k th complete ℓ -string.*

Proof. Calculating the matrix elements of the transfer matrix $A^{(\ell w; 0)}(\lambda) + D^{(\ell w; 0)}(\lambda)$ in the sector of M down-spins we show that there exists a pair of column vectors that are parallel to each other if $\lambda = \xi_k$. \square

Thus, the inverse matrix of the spin-1/2 transfer matrix $A^{(\ell w; 0)}(\lambda) + D^{(\ell w; 0)}(\lambda)$ does not exist at the special points. We remark that in the sector of $M = 0$ (and $M = L$), it is regular at $\lambda = \xi_k + n\pi\sqrt{-1}$ for $k = 1, 2, \dots, N_s$ and $n \in \mathbf{Z}$.

For an illustration, let us consider the case of $L = 2$ with $\ell = 1$ and $N_s = 1$. The operators A and D are explicitly given by

$$A_{12}^{(2+;0)}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{02} & c_{01}^+ c_{02}^- & 0 \\ 0 & 0 & b_{01} & 0 \\ 0 & 0 & 0 & b_{01} b_{02} \end{pmatrix}_{[1,2]}, \quad D_{12}^{(2+;0)}(\lambda) = \begin{pmatrix} b_{01} b_{02} & 0 & 0 & 0 \\ 0 & b_{01} & 0 & 0 \\ 0 & c_{01}^- c_{02}^+ & b_{02} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]}. \quad (3.41)$$

Here we have introduced b_{0j} and c_{0j}^\pm for $j = 1, 2$ by $b_{0j} = b(\lambda - w_j^{(2)})$ and $c_{0j}^\pm = \exp(\pm(\lambda - w_j^{(2)}))c(\lambda - w_j^{(2)})$ for $j = 1, 2$, respectively. Putting $\lambda = w_1^{(2)} = \xi_1$ we have

$$A_{12}^{(2+;0)}(\xi_1) + D_{12}^{(2+;0)}(\xi_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{[2]_q} & \frac{q^{-1}}{[2]_q} & 0 \\ 0 & \frac{q}{[2]_q} & \frac{1}{[2]_q} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]}. \quad (3.42)$$

We thus show that the transfer matrix is non-regular at $\lambda = w_1^{(2)} = \xi_1$:

$$\det \left(A_{12}^{(2+;0)}(\xi_1) + D_{12}^{(2+;0)}(\xi_1) \right) = 0. \quad (3.43)$$

In the sector of $M = 1$ the determinant is given by

$$\det \left(A_{12}^{(2+;0)}(\lambda) + D_{12}^{(2+;0)}(\lambda) \right) \Big|_{M=1} = \frac{4 \sinh(\lambda - \xi_1)}{\sinh(\lambda - \xi_1 + 2\eta)} \quad (3.44)$$

For an illustration, we shall show in Appendix C that there exists a pair of column vectors that are parallel to each other if we set $\lambda = \xi_k$, in the sector of $M = 1$, for the case of $L = 3$ with $w_1 = w_1^{(2)}$, $w_1 = w_2^{(2)}$ and $w_3 = \xi_2$.

Consequently, the QISP formula does not hold in the straightforward form for the operator-valued matrix elements of the monodromy matrix $T^{(\ell w;0)}(\lambda)$ for $w = \pm, p$ at $\lambda = w_{\ell(k-1)+1}^{(\ell)}$ for $k = 1, 2, \dots, N_s$. Here we recall that the monodromy matrix $T^{(\ell w;0)}(\lambda)$ is given by the spin-1/2 monodromy matrix $T^{(\ell w;\epsilon)}(\lambda)$ by putting $\epsilon = 0$.

4 Reduction of the matrix elements of spin- $\ell/2$ operators

4.1 Definition of the spin- $\ell/2$ off-shell matrix elements and the spin- $\ell/2$ form factors

Let $|0\rangle$ be the vacuum vector of the spin-1/2 chain of L sites; i.e., $|0\rangle = |\uparrow\rangle_1 \otimes \dots \otimes |\uparrow\rangle_L$. Here we recall that the symbol $\{\lambda_\alpha\}_M$ denotes a set of M parameters λ_α for $\alpha = 1, 2, \dots, M$.

For given sets of parameters $\{\mu_\alpha\}_N$ and $\{\lambda_\beta\}_M$ we define the off-shell Bethe covectors and vectors $\langle \{\mu_\alpha\}_N^{(\ell w)} |$ and $|\{\lambda_\beta\}_M^{(\ell w)}\rangle$, respectively, for $w = \pm, p$ as follows:

$$\langle \{\mu_\alpha\}_N^{(\ell w)} | = \langle 0 | \prod_{\alpha=1}^N C^{(\ell w)}(\mu_\alpha), \quad |\{\lambda_\beta\}_M^{(\ell w)}\rangle = \prod_{\beta=1}^M B^{(\ell w)}(\lambda_\beta) |0\rangle. \quad (4.1)$$

Here, parameters $\{\mu_\alpha\}_N$ and $\{\lambda_\beta\}_M$ do not necessarily satisfy the Bethe-ansatz equations. We define the spin- $\ell/2$ off-shell matrix elements of $E_k^{i_k, j_k^{(\ell w)}}$ for $w = \pm, p$ by

$$M_k^{i_k, j_k^{(\ell w)}}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) = \langle \{\mu_\alpha\}_N^{(\ell w)} | E_k^{i_k, j_k^{(\ell w)}} | \{\lambda_\beta\}_M^{(\ell w)} \rangle. \quad (4.2)$$

We also define the spin- $\ell/2$ off-shell matrix elements of the general elementary operators $\widehat{E}_k^{i_k, j_1^{(\ell w)}}$ for $w = \pm, p$ by

$$\widehat{M}_k^{i_k, j_1^{(\ell w)}}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) = \langle \{\mu_\alpha\}_N^{(\ell w)} | \widehat{E}_k^{i_k, j_1^{(\ell w)}} | \{\lambda_\beta\}_M^{(\ell w)} \rangle. \quad (4.3)$$

Let us assume that $\{\mu_\alpha\}_N$ and $\{\lambda_\beta\}_M$ satisfy the Bethe ansatz equations. We call $\langle \{\mu_\alpha\}_N^{(\ell w)} |$ and $|\{\lambda_\beta\}_M^{(\ell w)}\rangle$ the on-shell Bethe covectors and vectors, respectively. We define the spin- $\ell/2$ form factors of $E_k^{i_k, j_k^{(\ell w)}}$ for $w = \pm, p$ by

$$F_k^{i_k, j_k^{(\ell w)}}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) = \langle \{\mu_\alpha\}_N^{(\ell w)} | E_k^{i_k, j_k^{(\ell w)}} | \{\lambda_\beta\}_M^{(\ell w)} \rangle. \quad (4.4)$$

We also define the spin- $\ell/2$ form factors of the general elementary operators $\widehat{E}_k^{i_k, j_1(\ell w)}$ for $w = \pm, p$ by

$$\widehat{F}_k^{i_k, j_k(\ell w)}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) = \langle \{\mu_\alpha\}_N^{(\ell w)} | \widehat{E}_k^{i_k, j_k(\ell w)} | \{\lambda_\beta\}_M^{(\ell w)} \rangle. \quad (4.5)$$

We have defined the form factors of a local operator by the matrix elements of the operator between all pairs of the Bethe eigenvectors, in the paper. However, it is often the case that only the matrix elements between the ground state and excited states are called form factors.

4.2 Commutation relation with the projection operator

Lemma 4.1. *If spectral parameter λ is distinct from discrete values such as $w_j^{(\ell)} - \eta + n\pi\sqrt{-1}$ for $j = 1, 2, \dots, L$ and $n \in \mathbf{Z}$, the projection operator $P_{12\dots L}^{(\ell)}$ commutes with the matrix elements of the monodromy matrix $T_{0,12\dots L}^{(\ell+;0)}(\lambda) = T_{0,12\dots L}^{(1+)}(\lambda; \{w_j^{(\ell)}\}_L)$ as follows.*

$$P_{12\dots L}^{(\ell)} T_{0,12\dots L}^{(1+)}(\lambda; \{w_j^{(\ell)}\}_L) P_{12\dots L}^{(\ell)} = P_{12\dots L}^{(\ell)} T_{0,12\dots L}^{(1+)}(\lambda; \{w_j^{(\ell)}\}_L). \quad (4.6)$$

Let us assume that all the parameters in $\{\mu_\alpha\}_N$ and $\{\lambda_\beta\}_M$ are different from the discrete values given by $w_j^{(\ell)} - \eta + n\pi\sqrt{-1}$ for $j = 1, 2, \dots, L$ and $n \in \mathbf{Z}$. Here we recall that they correspond to N_s pieces of complete ℓ -strings minus η modulo $\pi\sqrt{-1}$, and the transfer matrix is singular at these points. Applying Lemma 4.1 we have

$$|\{\lambda_\beta\}_M^{(\ell+)}\rangle = \prod_{\beta=1}^M \left(P_{12\dots L}^{(\ell)} B^{(\ell+;0)}(\lambda_\beta) P_{12\dots L}^{(\ell)} \right) |0\rangle = P_{12\dots L}^{(\ell)} \prod_{\beta=1}^M B^{(\ell+;0)}(\lambda_\beta) |0\rangle \quad (4.7)$$

and

$$\langle \{\mu_\alpha\}_N^{(\ell+)} | = \langle 0 | \prod_{\alpha=1}^N \left(P_{12\dots L}^{(\ell)} C^{(\ell+;0)}(\mu_\alpha) P_{12\dots L}^{(\ell)} \right) = \langle 0 | \prod_{\alpha=1}^N C^{(\ell+;0)}(\mu_\alpha). \quad (4.8)$$

Here we remark that for the off-shell Bethe covectors we can absorb the projection operator acting to the left as follows.

$$\langle 0 | \prod_{\alpha=1}^N C^{(\ell+;0)}(\mu_\alpha) \cdot P_{12\dots L}^{(\ell)} = \langle 0 | \prod_{\alpha=1}^N C^{(\ell+;0)}(\mu_\alpha). \quad (4.9)$$

However, in eq. (4.7) we can not remove the projection operator acting to the right.

It follows from (4.7) and (4.8) that we can evaluate the spin- $\ell/2$ off-shell matrix elements by calculating the spin-1/2 ones. For instance, applying (4.7) and (4.8) we reduce every spin- $\ell/2$ off-shell matrix element into a spin-1/2 off-shell matrix element as follows.

$$\begin{aligned} \widehat{M}_k^{i, j(\ell+)}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) &= \langle 0 | \prod_{\alpha=1}^N C^{(\ell+;0)}(\mu_\alpha) \widehat{E}_k^{i, j(\ell+)} P_{12\dots L}^{(\ell)} \prod_{\beta=1}^M B^{(\ell+;0)}(\lambda_\beta) |0\rangle \\ &= \langle 0 | \prod_{\alpha=1}^N C^{(\ell+;0)}(\mu_\alpha) \widehat{E}_k^{i, j(\ell+)} \prod_{\beta=1}^M B^{(\ell+;0)}(\lambda_\beta) |0\rangle. \end{aligned} \quad (4.10)$$

Here we have made use of Lemma 3.1 in order to delete the projection operator.

We remark that in Refs. [12, 13, 14] there was a nontrivial assumption that the projection operator should commute with the operator-valued matrix elements of the spin-1/2 monodromy matrix $T^{(\ell, w; 0)}(\lambda)$ at an *arbitrary* value of the spectral parameter λ . In fact, the spin-1/2 monodromy matrix becomes singular if λ is equal to some discrete values such as $w_j - \eta$. At $\lambda = w_j - \eta$ the commutation relation of the monodromy matrix with the projection operator becomes non-trivial. If we multiply it with normalization factor $\sinh(\lambda - w_j + \eta)$ and define the normalized monodromy matrix, then its commutation relation with the projection operator becomes valid at $\lambda = w_j - \eta$.

4.3 Reduction of the spin- $\ell/2$ off-shell matrix elements into the spin-1/2 ones

For homogeneous gradings with $w = \pm$ and principal grading with $w = p$, we define $\sigma(w)$ by

$$\sigma(w) = \begin{cases} \pm 1 & \text{for } w = \pm, \\ 0 & \text{for } w = p. \end{cases} \quad (4.11)$$

We denote by \mathcal{S}_n the symmetric group of n elements.

Proposition 4.2. *Let i_1 and j_1 be integers satisfying $1 \leq i_1, j_1 \leq \ell$. For arbitrary parameters $\{\mu_\alpha\}_N$ and $\{\lambda_\beta\}_M$ with $i_1 - j_1 = N - M$ we have*

$$\begin{aligned} \widehat{M}_1^{i_1, j_1 (\ell w)}(\{\mu_k\}_N, \{\lambda_\beta\}_M) &= \langle 0 | \prod_{k=1}^N C^{(\ell w)}(\mu_k) \cdot \widehat{E}_1^{i_1, j_1 (\ell w)} \cdot \prod_{\beta=1}^M B^{(\ell w)}(\lambda_\beta) | 0 \rangle \\ &= \widehat{N}_{i_1, j_1}^{(\ell)} e^{\sigma(w)(\sum_k \mu_k - \sum_\gamma \lambda_\gamma)} \sum_{(\varepsilon_\beta(j_1))_\ell} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot e_1^{\varepsilon'_1(i_1), \varepsilon_1(j_1)} \dots e_\ell^{\varepsilon'_\ell(i_1), \varepsilon_\ell(j_1)} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle. \end{aligned} \quad (4.12)$$

Each summand is symmetric with respect to exchange of $\varepsilon'_\alpha(i_1)$; i.e., the following expression is independent of any permutation $\pi \in \mathcal{S}_\ell$:

$$\langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot e_1^{\varepsilon'_{\pi 1}(i_1), \varepsilon_1(j_1)} \dots e_\ell^{\varepsilon'_{\pi \ell}(i_1), \varepsilon_\ell(j_1)} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle. \quad (4.13)$$

Proof. In the case of homogeneous grading with $w = +$, we put (3.36) into (4.10), and we have (4.12) through the gauge transformation:

$$\begin{aligned} \langle 0 | \prod_{\alpha=1}^N C^{(\ell +; 0)}(\mu_\alpha) &= e^{\sum_\alpha \mu_\alpha} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot \chi_{12 \dots L}^{-1}, \\ \prod_{\beta=1}^M B^{(\ell +; 0)}(\lambda_\beta) | 0 \rangle &= \chi_{12 \dots L} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle e^{-\sum_\beta \lambda_\beta}. \end{aligned} \quad (4.14)$$

In the case of principal grading with $w = p$ we shall show explicitly in Appendix D the following:

$$\begin{aligned} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p)}(\mu_\alpha) &= \langle 0 | \prod_{k=1}^N C^{(\ell p; 0)}(\mu_k) \cdot \chi_{1 \dots L}^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots N_s}^{(\ell)}, \\ \prod_{\alpha=1}^N B^{(\ell p)}(\lambda_\alpha) | 0 \rangle &= \left(\chi_{1 \dots N_s}^{(\ell)} \right)^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots L} \cdot \prod_{\alpha=1}^M B^{(\ell p; 0)}(\lambda_\alpha) | 0 \rangle \end{aligned} \quad (4.15)$$

We thus evaluate the form factor as follows.

$$\begin{aligned} F_1^{i_1, j_1 (\ell p)} \{ \mu_\alpha \}_N, \{ \lambda_\beta \}_M &= \langle 0 | \prod_{\alpha=1}^N C^{(\ell p)}(\mu_\alpha) \cdot E_1^{i_1, j_1 (\ell p)} \cdot \prod_{\beta=1}^M B^{(\ell p)}(\lambda_\beta) | 0 \rangle \\ &= \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot \chi_{1 \dots L}^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots N_s}^{(\ell)} \cdot E_1^{i_1, j_1 (\ell p)} \cdot \left(\chi_{1 \dots N_s}^{(\ell)} \right)^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots L} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle \\ &= \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot \chi_{1 \dots L}^{-1} E_1^{i_1, j_1 (\ell p)} \chi_{1 \dots L} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle e^{(i_1 - j_1)(\xi_1 - (\ell - 1)\eta/2)}. \end{aligned} \quad (4.16)$$

Here we remark that

$$\chi_{1 \dots N_s}^{(\ell)} \cdot E_1^{i_1, j_1 (\ell p)} \cdot \left(\chi_{1 \dots N_s}^{(\ell)} \right)^{-1} = E_1^{i_1, j_1 (\ell p)} e^{(i_1 - j_1)(\xi_1 - (\ell - 1)\eta/2)}. \quad (4.17)$$

Applying Proposition 3.7 we obtain eq. (4.12) for the case of $w = p$. \square

Corollary 4.3. *Let us take integers i_k and j_k satisfying $1 \leq i_k, j_k \leq \ell$ for $k = 1, 2, \dots, m$. For arbitrary sets of parameters $\{ \mu_\alpha \}_N$ and $\{ \lambda_\beta \}_M$ with $\sum_{k=1}^m i_k - \sum_{k=1}^m j_k = N - M$, we have the matrix element for the m th product of the spin- $\ell/2$ elementary operators with entries (i_k, j_k) ($1 \leq k \leq m$) associated with principal grading as follows:*

$$\begin{aligned} \langle 0 | \prod_{\alpha=1}^N C^{(\ell w)}(\mu_\alpha) \cdot \prod_{k=1}^m \widehat{E}_k^{i_k, j_k (\ell w)} \cdot \prod_{\beta=1}^M B^{(\ell w)}(\lambda_\beta) | 0 \rangle &= \prod_{k=1}^m N_{i_k, j_k}^{(\ell)} \cdot e^{\sigma(w)(\sum_k \mu_k - \sum_\gamma \lambda_\gamma)} \\ \times \sum_{(\varepsilon_\beta^{[1]}(j_1))_\ell} \cdots \sum_{(\varepsilon_\beta^{[m]}(j_m))_\ell} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot \prod_{k=1}^m \left(e_{\ell(k-1)+1}^{\varepsilon_1^{[k]'}(i_k), \varepsilon_1^{[k]}(j_k)} \cdots e_{\ell(k-1)+\ell}^{\varepsilon_\ell^{[k]'}(i_k), \varepsilon_\ell^{[k]}(j_k)} \right) \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle. \end{aligned} \quad (4.18)$$

The summand is symmetric with respect to exchange of $\varepsilon_\alpha^{[k]'}(i_k)$ s for each k of $1 \leq k \leq m$; i.e., the following expression is independent of any permutation $\pi^{[k]} \in \mathcal{S}_\ell$ for $k = 1, 2, \dots, m$:

$$\langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot \prod_{k=1}^m e_{\ell(k-1)+1}^{\varepsilon_{\pi^{[k]}1}^{[k]'}(i_k), \varepsilon_1^{[k]}(j_k)} \cdots e_{\ell(k-1)+\ell}^{\varepsilon_{\pi^{[k]}\ell}^{[k]'}(i_k), \varepsilon_\ell^{[k]}(j_k)} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle. \quad (4.19)$$

4.4 Consequence of the continuity assumption of the Bethe roots

We now consider the Bethe-ansatz equations for the integrable spin- $\ell/2$ XXZ spin chain with inhomogeneity parameters ξ_b for $b = 1, 2, \dots, N_s$:

$$\frac{a^{(\ell)}(\lambda_\alpha)}{d^{(\ell)}(\lambda_\alpha; \{\xi_k\}_{N_s})} = \prod_{\beta=1; \beta \neq \alpha}^M \frac{\sinh(\lambda_\alpha - \lambda_\beta + \eta)}{\sinh(\lambda_\alpha - \lambda_\beta - \eta)} \quad (\alpha = 1, 2, \dots, M). \quad (4.20)$$

Here we recall $L = \ell N_s$. For integers ℓ we have set $a^{(\ell)}(\lambda_\alpha) = 1$ and defined $d^{(\ell)}(\mu; \{\xi_k\}_{N_s})$ by

$$d^{(\ell)}(\mu; \{\xi_k\}_{N_s}) = \prod_{k=1}^{N_s} \frac{\sinh(\mu - \xi_k)}{\sinh(\mu - \xi_k + \ell\eta)}. \quad (4.21)$$

Let $\{\lambda_\gamma\}_M$ be a solution of the Bethe-ansatz equations of the spin- $\ell/2$ chain with inhomogeneity parameters $\{\xi_b\}_{N_s}$. Suppose that $\{\lambda_\beta(\epsilon)\}_M$ denotes a solution of the spin-1/2 Bethe-ansatz equations with inhomogeneity parameters w_j being given by the N_s sets of the almost complete ℓ -strings, $w_j^{(\ell; \epsilon)}$ for $j = 1, 2, \dots, L$. They satisfy the Bethe-ansatz equations for the spin-1/2 XXZ spin chain:

$$\frac{a(\lambda_\alpha(\epsilon))}{d(\lambda_\alpha(\epsilon); \{w_j^{(\ell; \epsilon)}\}_L)} = \prod_{\beta=1; \beta \neq \alpha}^M \frac{\sinh(\lambda_\alpha(\epsilon) - \lambda_\beta(\epsilon) + \eta)}{\sinh(\lambda_\alpha(\epsilon) - \lambda_\beta(\epsilon) - \eta)} \quad (\alpha = 1, 2, \dots, M). \quad (4.22)$$

Here we have defined $d(\mu; \{w_j\}_L)$ by

$$d(\mu; \{w_j\}_L) = \prod_{j=1}^L b(\mu - w_j) = \prod_{j=1}^L \frac{\sinh(\mu - w_j)}{\sinh(\mu - w_j + \eta)}. \quad (4.23)$$

Then, the Bethe-ansatz equations (4.22) for the spin-1/2 XXZ chain with $w_j = w_j^{(\ell; \epsilon)}$ ($1 \leq j \leq L$) become those of the spin- $\ell/2$ XXZ chain by sending ϵ to zero. Here we remark the following limit:

$$\lim_{\epsilon \rightarrow 0} d(\mu; \{w_j^{(\ell; \epsilon)}\}) = d^{(\ell)}(\mu; \{\xi_k\}_{N_s}). \quad (4.24)$$

Let us now assume that the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ approach the Bethe roots $\{\lambda_\beta\}_M$ continuously in the limit of sending ϵ to 0. It follows that each entry of the Bethe-ansatz eigenstate of the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ is continuous with respect to ϵ . For a set of arbitrary parameters $\{\mu_k\}_N$ we therefore have

$$\begin{aligned} & \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_\alpha) \cdot e_1^{\epsilon'_1, \epsilon_1} \dots e_\ell^{\epsilon'_\ell, \epsilon_\ell} \cdot \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle \\ &= \lim_{\epsilon \rightarrow 0} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; \epsilon)}(\mu_\alpha) \cdot e_1^{\epsilon'_1, \epsilon_1} \dots e_\ell^{\epsilon'_\ell, \epsilon_\ell} \cdot \prod_{\beta=1}^M B^{(\ell p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle. \end{aligned} \quad (4.25)$$

The inhomogeneity parameters $w_j = w_j^{(\ell; \epsilon)}$ for $j = 1, 2, \dots, L$ are generic since the small number ϵ takes generic values and parameters r_b^β are also generic. Putting $w_j = w_j^{(\ell; \epsilon)}$ in (3.38)

we have the QISP formula with suffix $(1\,w)$ replaced by $(\ell\,w; \epsilon)$ for local operator x_n . We thus have the following expressions for $b = 1, 2, \dots, N_s$:

$$\begin{aligned} e_{\ell(b-1)+1}^{\epsilon'_1, \epsilon_1} \cdots e_{\ell(b-1)+\ell}^{\epsilon'_\ell, \epsilon_\ell} &= \prod_{k=1}^{\ell(b-1)} \left(A^{(\ell\,w; \epsilon)}(w_k^{(\ell; \epsilon)}) + D^{(\ell\,w; \epsilon)}(w_k^{(\ell; \epsilon)}) \right) \\ &\times T_{\epsilon_1, \epsilon'_1}^{(\ell\,w; \epsilon)}(w_{\ell(b-1)+1}^{(\ell; \epsilon)}) \cdots T_{\epsilon_\ell, \epsilon'_\ell}^{(\ell\,w; \epsilon)}(w_{\ell(b-1)+\ell}^{(\ell; \epsilon)}) \prod_{k=1}^{\ell b} \left(A^{(\ell\,w; \epsilon)}(w_k^{(\ell; \epsilon)}) + D^{(\ell\,w; \epsilon)}(w_k^{(\ell; \epsilon)}) \right)^{-1} \end{aligned} \quad (4.26)$$

For instance in the case of $b = 1$, applying formula (4.26) of $w = p$ we have

$$\begin{aligned} &\langle 0 | \prod_{\alpha=1}^N C^{(\ell\,p; \epsilon)}(\mu_\alpha) \cdot e_1^{\epsilon'_1, \epsilon_1} \cdots e_\ell^{\epsilon'_\ell, \epsilon_\ell} \cdot \prod_{\beta=1}^M B^{(\ell\,p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle \\ &= \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \langle 0 | \prod_{\alpha=1}^N C^{(\ell\,p; \epsilon)}(\mu_\alpha) \cdot T_{\epsilon_1, \epsilon'_1}^{(\ell\,p; \epsilon)}(w_1^{(\ell; \epsilon)}) \cdots T_{\epsilon_\ell, \epsilon'_\ell}^{(\ell\,p; \epsilon)}(w_\ell^{(\ell; \epsilon)}) \cdot \prod_{\beta=1}^M B^{(\ell\,p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle. \end{aligned} \quad (4.27)$$

Proposition 4.4. *Let $\{\mu_k\}_N$ be a set of arbitrary parameters and $\{\lambda_\alpha\}_M$ a solution of the spin- $\ell/2$ Bethe-ansatz equations. We denote by $\{\lambda_\alpha(\epsilon)\}_M$ a solution of the Bethe-ansatz equations for the spin- $1/2$ XXZ chain whose inhomogeneity parameters w_j are given by the N_s pieces of the almost complete ℓ -strings: $w_j = w_j^{(\ell; \epsilon)}$ for $1 \leq j \leq L$. We assume that the set $\{\lambda_\alpha(\epsilon)\}_M$ approaches $\{\lambda_\alpha\}_M$ continuously when we send ϵ to zero. For the Bethe states $\langle \{\mu_k\}_N |$ and $| \{\lambda_\alpha\}_M \rangle$, which are off-shell and on-shell, respectively, we evaluate the matrix elements of a given product of elementary operators $e_1^{\epsilon'_1, \epsilon_1} \cdots e_\ell^{\epsilon'_\ell, \epsilon_\ell}$ as follows.*

$$\begin{aligned} &\langle 0 | \prod_{\alpha=1}^N C^{(\ell\,p; 0)}(\mu_\alpha) e_1^{\epsilon'_1, \epsilon_1} \cdots e_\ell^{\epsilon'_\ell, \epsilon_\ell} \prod_{\beta=1}^M B^{(\ell\,p; 0)}(\lambda_\beta) | 0 \rangle \\ &= \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \lim_{\epsilon \rightarrow 0} \langle 0 | \prod_{\alpha=1}^N C^{(\ell\,p; \epsilon)}(\mu_\alpha) T_{\epsilon_1, \epsilon'_1}^{(\ell\,p; \epsilon)}(w_1^{(\ell; \epsilon)}) \cdots T_{\epsilon_\ell, \epsilon'_\ell}^{(\ell\,p; \epsilon)}(w_\ell^{(\ell; \epsilon)}) \prod_{\beta=1}^M B^{(\ell\,p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle, \end{aligned} \quad (4.28)$$

where $\phi_m(\{\lambda_\beta\})$ has been defined by $\phi_m(\{\lambda_\beta\}; \{w_j\}) = \prod_{j=1}^m \prod_{\alpha=1}^M b(\lambda_\alpha - w_j)$ with $b(u) = \sinh(u)/\sinh(u + \eta)$.

4.5 Spin- $\ell/2$ form factors reduced into the spin- $1/2$ ones

Combining Propositions 4.2, and 4.4 we have the following:

Proposition 4.5. *Let i_1 and j_1 be integers satisfying $1 \leq i_1, j_1 \leq \ell$. We set $i_1 - j_1 = N - M$. Let $\{\mu_k\}_N$ be a set of arbitrary N parameters. For a set of Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ which approaches*

$\{\lambda_\beta\}_M$ continuously at $\epsilon = 0$ we have the following:

$$\begin{aligned}
& \langle 0 | \prod_{\alpha=1}^N C^{(\ell w)}(\mu_\alpha) \cdot \widehat{E}_1^{i_1, j_1 (\ell w)} \cdot \prod_{\beta=1}^M B^{(\ell w)}(\lambda_\beta) | 0 \rangle \\
&= \widehat{N}_{i_1, j_1}^{(\ell)} e^{\sigma(w)(\sum_k \mu_k - \sum_\gamma \lambda_\gamma)} \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \\
&\times \sum_{(\varepsilon_\beta(j_1))_\ell} \lim_{\epsilon \rightarrow 0} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; \epsilon)}(\mu_\alpha) T_{\varepsilon_1(j_1), \varepsilon'_1(i_1)}^{(\ell p; \epsilon)}(w_1^{(\ell; \epsilon)}) \cdots T_{\varepsilon_\ell(j_1), \varepsilon'_\ell(j_\ell)}^{(\ell p; \epsilon)}(w_\ell^{(\ell; \epsilon)}) \prod_{\beta=1}^M B^{(\ell p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle.
\end{aligned} \tag{4.29}$$

Let us recall a product of the general spin- $\ell/2$ elementary operators, $\widehat{E}_1^{i_1, j_1 (\ell w)} \cdots \widehat{E}_m^{i_m, j_m (\ell w)}$, which we have introduced in Corollary 3.9. We also recall variables $\varepsilon_\alpha^{[k]'}(i_k)$ and $\varepsilon_\beta^{[k]}(j_k)$ which take only two values 0 or 1 for $k = 1, 2, \dots, m$ and $\alpha, \beta = 0, 1, \dots, \ell$. We have the following:

Corollary 4.6. *Let us take integers i_k and j_k satisfying $1 \leq i_k, j_k \leq \ell$ for $k = 1, 2, \dots, m$. We set $\sum_k i_k - \sum_k j_k = N - M$. Let $\{\mu_k\}_N$ be a set of arbitrary N parameters. If the set of the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ approaches the set of the Bethe roots $\{\lambda_\beta\}_M$ continuously at $\epsilon = 0$, we have the following:*

$$\begin{aligned}
& \langle 0 | \prod_{\alpha=1}^N C^{(\ell w)}(\mu_\alpha) \cdot \prod_k \widehat{E}_k^{i_k, j_k (\ell w)} \cdot \prod_{\beta=1}^M B^{(\ell w)}(\lambda_\beta) | 0 \rangle \\
&= \left(\prod_{k=1}^m \widehat{N}_{i_k, j_k}^{(\ell_k)} \right) \cdot e^{\sigma(w)(\sum_{k=1}^N \mu_k - \sum_{\gamma=1}^M \lambda_\gamma)} \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \\
&\times \sum_{(\varepsilon_\beta^{[1]}(j_1))_\ell} \cdots \sum_{(\varepsilon_\beta^{[m]}(j_m))_\ell} \lim_{\epsilon \rightarrow 0} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; \epsilon)}(\mu_\alpha) \prod_{k=1}^m \left(T_{\varepsilon_1^{[1]}(j_k), \varepsilon_1^{[k]'}(i_k)}^{(\ell p; \epsilon)}(w_1^{(\ell; \epsilon)}) \cdots T_{\varepsilon_\ell(j_k), \varepsilon'_\ell(j_k)}^{(\ell p; \epsilon)}(w_\ell^{(\ell; \epsilon)}) \right) \\
&\times \prod_{\beta=1}^M B^{(\ell p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle.
\end{aligned} \tag{4.30}$$

Here we have chosen sequences $\varepsilon_\alpha^{[k]'}(j_k)$ for each integer k of $1 \leq k \leq m$.

5 Spin- $\ell/2$ form factors via the spin-1/2 scalar products

5.1 Fundamental commutation relations

For given sequences $(\varepsilon'_\alpha)_m$ and $(\varepsilon_\beta)_m$ we consider sets α^\pm defined by eqs. (3.10). We also denote the sets α^\pm by $\alpha^-(\{\varepsilon'_\alpha\})$ and $\alpha^+(\{\varepsilon'_\alpha\})$, respectively, in order to show their dependence on the sequences $(\varepsilon'_\alpha)_m$ and $(\varepsilon_\beta)_m$ explicitly. We take a set of distinct integers a_j for $j \in \alpha^-$ and a'_k for $k \in \alpha^+$ such that they satisfy $1 \leq a_j \leq N$ for $j \in \alpha^-$ and $1 \leq a'_k \leq N + k$ for $k \in \alpha^+$. For the given set of a_j, a'_j , we introduce \mathbf{A}_j and \mathbf{A}'_j by

$$\begin{aligned}
\mathbf{A}_j &= \{b; 1 \leq b \leq N + m, b \neq a_k, a'_k \text{ for } k < j\}, \\
\mathbf{A}'_j &= \{b; 1 \leq b \leq N + m, b \neq a_k \text{ for } k \leq j, b \neq a'_k \text{ for } k < j\}.
\end{aligned} \tag{5.1}$$

Setting rapidities μ_{N+j} by

$$\mu_{N+j} = w_j, \quad \text{for } j = 1, 2, \dots, m, \quad (5.2)$$

we can show the fundamental commutation relations as follows [7].

$$\begin{aligned} & \langle 0 | \left(\prod_{\alpha=1}^N C^{(1p)}(\mu_\alpha) \right) T_{\varepsilon_1, \varepsilon'_1}^{(1p)}(\mu_{N+1}) \cdots T_{\varepsilon_{2sm}, \varepsilon'_m}^{(1p)}(\mu_{N+m}) \\ &= \left(\prod_{j \in \alpha^-(\{\varepsilon'_\alpha\})} \sum_{a_j=1}^N \prod_{j \in \alpha^+(\{\varepsilon_\beta\})} \sum_{a'_j=1}^{N+j} \right) G_{\{a_j, a'_j\}}^{(\varepsilon'_\alpha)_m, (\varepsilon_\beta)_m}((\mu_k)_{N+m}) \langle 0 | \prod_{k \in \mathbf{A}_{m+1}(\{a_j, a'_j\})} C^{(1p)}(\mu_k), \end{aligned}$$

where coefficients $G_{\{a_j, a'_j\}}^{(\varepsilon'_\alpha)_m, (\varepsilon_\beta)_m}((\mu_k)_{N+m})$ are given by

$$\begin{aligned} G_{\{a_j, a'_j\}}^{(\varepsilon'_\alpha)_m, (\varepsilon_\beta)_m}((\mu_k)_{N+m}) &= \prod_{j \in \alpha^+(\{\varepsilon_\beta\})} \left(\frac{\prod_{b=1; b \in \mathbf{A}_j}^{N+j-1} \sinh(\mu_b - \mu_{a'_j} + \eta)}{\prod_{b=1, b \in \mathbf{A}_{j+1}}^{N+j} \sinh(\mu_b - \mu_{a'_j})} \right) \\ &\times \prod_{j \in \alpha^-(\{\varepsilon'_\alpha\})} \left(d(\mu_{a_j}; \{w_k\}_L) \frac{\prod_{b=1; b \in \mathbf{A}_j}^{N+j-1} \sinh(\mu_{a_j} - \mu_b + \eta)}{\prod_{b=1, b \in \mathbf{A}'_j}^{N+j} \sinh(\mu_{a_j} - \mu_b)} \right). \end{aligned} \quad (5.3)$$

Here we recall

$$d(\mu; \{w_k\}_L) = \prod_{k=1}^L b(\mu - w_k). \quad (5.4)$$

We consider the sums over integers a_j and a'_k such that they satisfy $1 \leq a_j \leq N$ and $1 \leq a'_k \leq N+j$ for $j \in \alpha^-$ and $k \in \alpha^+$, respectively. Hereafter, we express the products of the sums over a_j and a'_k by the symbol $\sum_{\{a_j, a'_j\}}$, as follows.

$$\sum_{\{a_j, a'_j\}} = \prod_{j \in \alpha^-} \left(\sum_{a_j=1}^N \right) \prod_{j \in \alpha^+} \left(\sum_{a'_j=1}^{N+j} \right). \quad (5.5)$$

5.2 Form factors as a sum of the spin-1/2 scalar products

We shall evaluate the spin- $\ell/2$ form factor $\widehat{F}_k^{i,j(\ell w)}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M)$.

We first define the scalar product in the spin-1/2 case for two sets of M parameters $\{\mu_k\}_M$ and $\{\lambda_\gamma\}_M$ by

$$S_M^{(1)}(\{\mu_1, \dots, \mu_M\}, \{\lambda_1, \dots, \lambda_M\}; \{w_j\}_L) = \langle 0 | \prod_{k=1}^M C^{(1p)}(\mu_k) \prod_{\gamma=1}^M B^{(1p)}(\lambda_\gamma) | 0 \rangle. \quad (5.6)$$

Here $\{\mu_k\}_M$ and $\{\lambda_\gamma\}_M$ are not necessarily solutions of the Bethe ansatz equations. We define the scalar product for the spin- $\ell/2$ operators $B^{(\ell p)}(\mu_k)$ and $C^{(\ell p)}(\lambda_k)$ for $k = 1, 2, \dots, M$, by

$$S_M^{(\ell)}(\{\mu_\alpha\}, \{\lambda_\beta\}; \{\xi_k\}_{N_s}) = \langle 0 | \prod_{\alpha=1}^M C^{(\ell p)}(\mu_\alpha) \prod_{\beta=1}^M B^{(\ell p)}(\lambda_\beta) | 0 \rangle. \quad (5.7)$$

Here we also recall that $\{\mu_k\}_M$ and $\{\lambda_\gamma\}_M$ are not necessarily Bethe roots.

Let us first review Slavnov's formula of scalar products in the spin-1/2 case [2]: if $\lambda_1, \lambda_2, \dots, \lambda_M$ satisfy the spin-1/2 Bethe-ansatz equations for the spin-1/2 XXZ spin chain with inhomogeneity parameters w_j , the scalar product is expressed in terms of the determinant:

$$S_M^{(1)}(\{\mu_\alpha\}, \{\lambda_\beta\}; \{w_j\}_L) = \frac{\det \hat{H}(\{\lambda_\alpha\}_M, \{\mu_k\}_M; \{w_j\}_L)}{\prod_{1 \leq j < k \leq M} \sinh(\mu_j - \mu_k) \prod_{1 \leq \alpha < \beta \leq M} \sinh(\lambda_\beta - \lambda_\alpha)}. \quad (5.8)$$

Here, the matrix elements of \hat{H} with entry (a, b) for $a, b = 1, 2, \dots, M$ are given by

$$\begin{aligned} \hat{H}_{a,b}(\{\lambda\}_M, \mu_b; \{w_j\}_L) &= \frac{\sinh \eta}{\sinh(\lambda_a - \mu_b)} \left(a(\mu_b) \prod_{k=1; k \neq a}^M \sinh(\lambda_k - \mu_b + \eta) \right. \\ &\quad \left. - d(\mu_b; \{w_j\}_L) \prod_{k=1; k \neq a}^M \sinh(\lambda_k - \mu_b - \eta) \right). \end{aligned} \quad (5.9)$$

Here we recall that $d(\mu; \{w_j\}_L) = \prod_{j=1}^L \sinh(\mu - w_j) / \sinh(\mu - w_j + \eta)$.

We remark that it is sometimes useful to make use of the following relations:

Lemma 5.1. *For two sets of arbitrary parameters $\{\mu_k\}_{N+m}$ and $\{\lambda_\gamma\}_M$ we have*

$$\begin{aligned} &\langle 0 | \prod_{\alpha=1}^N C^{(1p)}(\mu_\alpha) \cdot T_{\varepsilon_1, \varepsilon'_1}^{(1p)}(\mu_{N+1}) \cdots T_{\varepsilon_m, \varepsilon'_m}^{(1p)}(\mu_{N+m}) \cdot \prod_{\gamma=1}^M B^{(1p)}(\lambda_\gamma) | 0 \rangle \\ &= \langle 0 | \prod_{\beta=1}^M C^{(1p)}(\lambda_\beta) \cdot T_{\varepsilon'_m, \varepsilon_m}^{(1p)}(\mu_{N+m}) \cdots T_{\varepsilon'_1, \varepsilon_1}^{(1p)}(\mu_{N+1}) \cdot \prod_{\gamma=1}^M B^{(1p)}(\mu_\gamma) | 0 \rangle. \end{aligned} \quad (5.10)$$

We now consider the matrix element of an m th product of the spin-1/2 operators with respect to given bra and ket vectors, $\langle \{\mu_\alpha\}_N^{(\ell p; 0)} |$ and $| \{\lambda_\alpha\}_N^{(\ell p; 0)} \rangle$, respectively. We define P by $P = N - M$, where P can be negative. Setting $\mu_{N+j} = w_j$ for $j = 1, 2, \dots, m$, we have

$$\begin{aligned} &\langle 0 | \left(\prod_{\alpha=1}^N C^{(1p)}(\mu_\alpha) \right) T_{\varepsilon_1, \varepsilon'_1}^{(1p)}(\mu_{N+1}) \cdots T_{\varepsilon_m, \varepsilon'_m}^{(1p)}(\mu_{N+m}) \cdot \prod_{\gamma=1}^M B^{(1p)}(\lambda_\gamma) | 0 \rangle \\ &= \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\varepsilon'_\alpha)_m, (\varepsilon_\beta)_m}((\mu_k)_{N+m}) \langle 0 | \prod_{k \in \mathbf{A}_{m+1}(\{a_j, a'_j\})} C(\mu_k) \cdot \prod_{\gamma=1}^M B(\lambda_\gamma) | 0 \rangle \\ &= \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\varepsilon'_\alpha)_m, (\varepsilon_\beta)_m}((\mu_k)_{N+m}) \\ &\quad \times S_M^{(1)}(\{\mu_1, \dots, \mu_N, w_1, \dots, w_m\} \setminus \{\mu_{a_j}, \mu_{a'_k}\}_{m+P}, \{\lambda_\gamma\}_M; \{w_j\}_L). \end{aligned} \quad (5.11)$$

Here we remark that the number of elements in the set $\{a_j, a'_k\}$ is given by $m + P$.

Let us now consider the case of $m = \ell$ for the form factor of the spin- $\ell/2$ operators. For a given sequence of parameters μ_k for $1 \leq k \leq N$ we extend it into a sequence of length $N + \ell$ by setting $\mu_k = w_{k-N}^{(\ell)}$ for $k = N + 1, \dots, N + \ell$. We define another sequence $\mu_k(\epsilon)$ for $k = 1, 2, \dots, N + \ell$ by

$$\mu_k(\epsilon) = \begin{cases} \mu_k & \text{for } 1 \leq k \leq N, \\ w_{k-N}^{(\ell; \epsilon)} & \text{for } N < k \leq N + \ell. \end{cases} \quad (5.12)$$

Substituting (5.11) into (4.28) we have

$$\begin{aligned} & \langle 0 | \prod_{\alpha=1}^N C^{(\ell p; 0)}(\mu_a) e_1^{\epsilon'_1, \epsilon_1} \dots e_\ell^{\epsilon'_\ell, \epsilon_\ell} \prod_{\beta=1}^M B^{(\ell p; 0)}(\lambda_\beta) | 0 \rangle \\ &= \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\epsilon'_\alpha)_\ell, (\epsilon_\beta)_\ell}((\mu_k)_{N+\ell}) \\ & \times \lim_{\epsilon \rightarrow 0} S_M^{(1)}(\{\mu_1(\epsilon), \dots, \mu_N(\epsilon), w_1^{(\ell; \epsilon)}, \dots, w_\ell^{(\ell; \epsilon)}\} \setminus \{\mu_{a_j}(\epsilon), \mu_{a'_k}(\epsilon)\}_{\ell+P}, \{\lambda_\gamma(\epsilon)\}_M; \{w_j^{(\ell; \epsilon)}\}_L). \end{aligned} \quad (5.13)$$

We shall explicitly express the limiting procedure in the last line of (5.13). Here we recall the Bethe-ansatz equations (BAE) for the spin- $\ell/2$ case (4.20) and the limiting procedure (4.24), where the spin- $\ell/2$ BAE is derived from the spin-1/2 BAE with almost complete strings $w_j^{(\ell; \epsilon)}$ by sending ϵ to 0. Let us now consider the case where some of μ_k are given by inhomogeneity parameters w_j . We first define the following function:

$$d^{(\ell)'}(\mu; \{\xi_k\}_{N_s}) = \begin{cases} \prod_{k=1}^{N_s} \frac{\sinh(\mu - \xi_k)}{\sinh(\mu - \xi_k + \eta)} & \text{for } \mu \neq w_j^{(\ell)}, \\ 0 & \text{for } \mu = w_j^{(\ell)}. \end{cases} \quad (5.14)$$

We next introduce the matrix $\widehat{H}^{(\ell)'}$. We define the matrix elements of the matrix $\widehat{H}^{(\ell)'}$ with entry (a, b) for $a, b = 1, 2, \dots, M$, by

$$\begin{aligned} \widehat{H}_{a,b}^{(\ell)'}(\{\lambda\}_M, \mu_b; \{\xi_k\}_{N_s}) &= \frac{\sinh \eta}{\sinh(\lambda_a - \mu_b)} \left(a(\mu_b) \prod_{k=1; k \neq a}^M \sinh(\lambda_k - \mu_b + \eta) \right. \\ & \quad \left. - d^{(\ell)'}(\mu_b; \{\xi_k\}_{N_s}) \prod_{k=1; k \neq a}^M \sinh(\lambda_k - \mu_b - \eta) \right). \end{aligned} \quad (5.15)$$

If $\{\lambda_\beta\}_M$ satisfy the Bethe ansatz equations of the spin- $\ell/2$ XXZ spin chain with inhomogeneity parameters ξ_k , we define the expression $S_M^{(\ell)'}(\{\mu_k\}, \{\lambda\}; \{\xi_k\}_{N_s})$ by

$$S_M^{(\ell)'}(\{\mu_k\}_M, \{\lambda\}_M; \{\xi_k\}_{N_s}) = \frac{\det \widehat{H}^{(\ell)'}(\{\lambda_\alpha\}_M, \{\mu_k\}_M; \{\xi_k\}_{N_s})}{\prod_{1 \leq j < k \leq M} \sinh(\mu_j - \mu_k) \prod_{1 \leq \alpha < \beta \leq M} \sinh(\lambda_\beta - \lambda_\alpha)}. \quad (5.16)$$

In eq.(5.13), sending ϵ to zero, for a given set of integers $\{a_j, a'_k\}$ we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} S_M^{(1)}(\{\mu_k(\epsilon)\}_{N+\ell} \setminus \{\mu_{a_j}(\epsilon), \mu_{a'_k}(\epsilon)\}_{\ell+P}, \{\lambda_\beta(\epsilon)\}_M; \{w_j^{(\ell; \epsilon)}\}_L) \\ &= S_M^{(\ell)'}(\{\mu_k\}_{N+\ell} \setminus \{\mu_{a_j}, \mu_{a'_k}\}_{\ell+P}, \{\lambda_\beta\}_M; \{\xi_k\}_{N_s}). \end{aligned} \quad (5.17)$$

We summarize the result as follows:

Lemma 5.2. *Let $\{\lambda_\gamma\}_M$ be a solution of the Bethe ansatz equations for the spin- $\ell/2$ chain with inhomogeneity parameters ξ_k ($1 \leq k \leq N_s$), and $\{\mu_k\}_N$ a set of arbitrary parameters. We assume that $\{\lambda_\beta(\epsilon)\}$ is a solution of the Bethe ansatz equations for the spin-1/2 chain with $w_j = w_j^{(\ell; \epsilon)}$ ($1 \leq j \leq L$) and it approaches $\{\lambda_\gamma\}_M$ continuously at $\epsilon = 0$. We express the matrix elements of a product of the spin-1/2 operators in the limit of sending ϵ to 0 in terms of the modified scalar product $S^{(\ell)'}$ as follows.*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \langle 0 | \prod_{k=1}^N C^{(\ell p; \epsilon)}(\mu_k) T_{\epsilon_1, \epsilon'_1}^{(\ell p; \epsilon)}(w_1^{(\ell; \epsilon)}) \cdots T_{\epsilon_\ell, \epsilon'_\ell}^{(\ell p; \epsilon)}(w_\ell^{(\ell; \epsilon)}) \prod_{\beta=1}^M B^{(\ell p; \epsilon)}(\lambda_\beta(\epsilon)) | 0 \rangle \\ &= \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\epsilon'_\alpha)_\ell, (\epsilon_\beta)_\ell}((\mu_k)_{N+\ell}) \\ & \quad \times S_M^{(\ell)'}(\{\mu_1, \dots, \mu_N, w_1^{(\ell)}, \dots, w_\ell^{(\ell)}\} \setminus \{\mu_{a_j}, \mu_{a'_k}\}_{\ell+P}, \{\lambda_\gamma\}_M; \{\xi_k\}_{N_s}). \end{aligned} \quad (5.18)$$

Here, $P = N - M$ and we have set $\mu_{N+j} = w_j^{(\ell)}$ for $j = 1, 2, \dots, \ell$.

Through Lemma 5.2 we show the following:

Proposition 5.3. *Let $\{\lambda_\gamma\}_M$ be a solution of the Bethe-ansatz equations for the spin- $\ell/2$ chain with inhomogeneity parameters ξ_k ($1 \leq k \leq N_s$), and $\{\mu_k\}_N$ a set of arbitrary parameters. We assume that $\{\lambda_\beta(\epsilon)\}$ is a solution of the Bethe ansatz equations for the spin-1/2 chain with $w_j = w_j^{(\ell; \epsilon)}$ ($1 \leq j \leq L$). Then, every spin- $\ell/2$ form factor associated with the Bethe roots $\{\lambda_\gamma\}_M$ can be expressed as a sum of the spin-1/2 scalar products. For instance, we evaluate the form factor of the spin- $\ell/2$ elementary operator as follows.*

$$\begin{aligned} & \widehat{F}_{k=1}^{i_1, j_1 (\ell w)}(\{\mu_\alpha\}_N, \{\lambda_\beta\}_M) = \langle 0 | \prod_{\alpha=1}^N C^{(\ell w)}(\mu_\alpha) \cdot \widehat{E}_1^{i_1, j_1 (\ell w)} \cdot \prod_{\beta=1}^M B^{(\ell w)}(\lambda_\beta) | 0 \rangle \\ &= \widehat{N}_{i_1, j_1}^{(\ell)} e^{\sigma(w)(\sum_k \mu_k - \sum_\gamma \lambda_\gamma)} \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \sum_{(\epsilon_\beta(j_1))_\ell} \sum_{\{a_j, a'_k\}} G_{\{a_j, a'_k\}}^{(\epsilon'_\alpha(i_1))_\ell, (\epsilon_\beta(j_1))_\ell}((\mu_k)_{N+\ell}) \\ & \quad \times S_M^{(\ell)'}(\{\mu_1, \dots, \mu_N, w_1^{(\ell)}, \dots, w_\ell^{(\ell)}\} \setminus \{\mu_{a_j}, \mu_{a'_k}\}_{\ell+P}, \{\lambda_\gamma\}_M; \{\xi_k\}_{N_s}). \end{aligned} \quad (5.19)$$

Here we have fixed a sequence $\epsilon'_\alpha(i_1)$.

It is easy to show the formula corresponding to (5.19) for a given product of the spin- $\ell/2$ elementary operators such as $\widehat{E}_1^{i_1, j_1 (\ell w)} \widehat{E}_2^{i_2, j_2 (\ell w)} \cdots \widehat{E}_m^{i_m, j_m (\ell w)}$.

6 Spin- s XXZ correlation functions in a massless region

Applying the reduction formula we now derive the multiple-integral representations of the correlation functions of the integrable spin- s XXZ spin chain in a region of the massless regime:

$0 \leq \zeta < \pi/2s$. Here we remark that integer $2s$ corresponds to integer ℓ of $V^{(\ell)}$. We show only the main results. In fact, we derive them by following mainly the procedures of Ref. [13] except for the evaluation of the expectation values of products of the spin- s operators.

Let us review the main procedures for deriving the multiple-integral representation of the spin- s XXZ correlation functions, briefly. First, we introduce the spin- s elementary operators as the basic blocks for constructing the local operators of the integrable spin- s XXZ spin chain. Secondary, we reduce them into a sum of products of the spin-1/2 elementary operators, which we express in terms of the matrix elements of the spin-1/2 monodromy matrix through the spin-1/2 QISP formula. We then evaluate their scalar products with Slavnov's formula of the Bethe-ansatz scalar products, we have shown in §5. Here, the expectation value of a physical quantity is expressed as a sum of the ratios of the Bethe-ansatz scalar products to the norm of the ground-state Bethe-ansatz eigenvector, and the ratios are expressed in terms of the determinants of some matrices. Thirdly, by solving the integral equations for the matrices in the thermodynamic limit, we derive the multiple-integral representation of the correlation functions. Here, the integrals and their solutions for the spin- s case are given in Ref. [13].

6.1 Conjecture of the spin- s Ground-state solution

Let us now consider the ground state of the integrable spin- s XXZ spin chain in the massless regime. Here we remark that integer $2s$ corresponds to integer ℓ of $V^{(\ell)}$. In the massless regime we set $\eta = i\zeta$ with $0 \leq \zeta < \pi$. For the spin- s case, in the region $0 \leq \zeta < \pi/2s$ we assume that the spin- s ground state $|\psi_g^{(2s)}\rangle$ is given by $N_s/2$ sets of the $2s$ -strings:

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \delta_a^{(\alpha)}, \quad \text{for } a = 1, 2, \dots, N_s/2 \text{ and } \alpha = 1, 2, \dots, 2s. \quad (6.1)$$

Here we also assume that string deviations $\delta_a^{(\alpha)}$ are small enough when N_s is large enough. In terms of $\lambda_a^{(\alpha)}$, the spin- s ground state associated with grading w is given by

$$|\psi_g^{(2s w)}\rangle = \prod_{a=1}^{N_s/2} \prod_{\alpha=1}^{2s} B^{(2s w)}(\lambda_a^{(\alpha)}; \{\xi_b\}_{N_s}) |0\rangle. \quad (6.2)$$

Here we have M Bethe roots with $M = 2s N_s/2 = sN_s$.

6.2 Multiple-integral representations for arbitrary matrix elements

Let us now formulate the multiple-integral representations of the spin- s XXZ correlation functions in the general case for the massless region: $0 \leq \zeta < \pi/2s$. We define the zero-temperature correlation function for a given product of the general spin- s elementary operators with grading w $\widehat{E}_1^{i_1, j_1 (2s p)} \dots \widehat{E}_m^{i_m, j_m (2s p)}$, which are $(2s+1) \times (2s+1)$ matrices, by

$$\widehat{F}_m^{(2s p)}(\{i_k, j_k\}) = \langle \psi_g^{(2s w)} | \prod_{k=1}^m \widehat{E}_k^{i_k, j_k (2s w)} | \psi_g^{(2s w)} \rangle / \langle \psi_g^{(2s w)} | \psi_g^{(2s w)} \rangle. \quad (6.3)$$

For the m th product of elementary operators, we introduce the sets of variables $\varepsilon_\alpha^{[k]}'$ s and $\varepsilon_\beta^{[k]}$ s ($1 \leq k \leq m$) such that the number of $\varepsilon_\alpha^{[k]}' = 1$ with $1 \leq \alpha \leq 2s$ is given by i_k and the number of $\varepsilon_\beta^{[k]} = 1$ with $1 \leq \beta \leq 2s$ by j_k , respectively. Here, the variables $\varepsilon_\alpha^{[k]}'$ and $\varepsilon_\beta^{[k]}$ take only two values 0 or 1 (see also Corollary 3.9). We then express them by integers ε_j' s and ε_j s for $j = 1, 2, \dots, 2sm$ as follows:

$$\begin{aligned}\varepsilon_{2s(k-1)+\alpha}' &= \varepsilon_\alpha^{[k]}' \quad \text{for } \alpha = 1, 2, \dots, 2s; k = 1, 2, \dots, m, \\ \varepsilon_{2s(k-1)+\beta} &= \varepsilon_\beta^{[k]} \quad \text{for } \beta = 1, 2, \dots, 2s; k = 1, 2, \dots, m.\end{aligned}\tag{6.4}$$

For given sets of ε_j and ε_j' for $j = 1, 2, \dots, 2sm$ we define α^- by the set of integers j satisfying $\varepsilon_j' = 1$ and α^+ by the set of integers j satisfying $\varepsilon_j = 0$:

$$\alpha^-(\{\varepsilon_j'\}) = \{j; \varepsilon_j' = 1\}, \quad \alpha^+(\{\varepsilon_j\}) = \{j; \varepsilon_j = 0\}.\tag{6.5}$$

We denote by r and r' the number of elements of the set α^- and α^+ , respectively. Due to charge conservation, we have $r + r' = 2sm$. Precisely, we have $r = \sum_{k=1}^m i_k$ and $r' = 2sm - \sum_{k=1}^m j_k$.

For sets α^- and α^+ , which correspond to $\{\varepsilon_a'\}$ and $\{\varepsilon_b\}$, respectively, we define integral variables $\tilde{\lambda}_j$ for $j \in \alpha^-$ and $\tilde{\lambda}_j'$ for $j \in \alpha^+$, respectively, by the following:

$$(\tilde{\lambda}_{j_{max}}', \dots, \tilde{\lambda}_{j_{min}}', \tilde{\lambda}_{j_{min}}, \tilde{\lambda}_{j_{max}}) = (\lambda_1, \dots, \lambda_{2sm}).\tag{6.6}$$

We now introduce a matrix $S = S((\lambda_j)_{2sm}; (w_j^{(2s)})_{2sm})$. For each integer j satisfying $1 \leq j \leq 2sm$, we define $\alpha(\lambda_j)$ by $\alpha(\lambda_j) = \gamma$ for an integer γ satisfying $1 \leq \gamma \leq 2s$ if λ_j is related to an integral variable μ_j through $\lambda_j = \mu_j - (\gamma - 1/2)\eta$ or if λ_j takes a value close to $w_k^{(2s)}$ with $\beta(k) = \gamma$. Thus, μ_j corresponds to the “string center” of λ_j . Here we have defined $\beta(j)$ by

$$\beta(j) = j - 2s[[(j-1)/2s]] \quad (1 \leq j \leq M).\tag{6.7}$$

Here $[[x]]$ denotes the greatest integer less than or equal to x . We define the (j, k) element of the matrix S by

$$S_{j,k} = \rho(\lambda_j - w_k^{(2s)} + \eta/2) \delta(\alpha(\lambda_j), \beta(k)), \quad \text{for } j, k = 1, 2, \dots, 2sm.\tag{6.8}$$

Here $\rho(\lambda)$ denotes the density of string centers [13], and $\delta(\alpha, \beta)$ the Kronecker delta. We obtain the following multiple-integral representation:

$$\begin{aligned}\widehat{F}_m^{(2s w)}(\{i_k, j_k\}) &= \widehat{C}^{(2s)}(\{i_k, j_k\}) \times \\ &\times \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \dots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_1 \dots \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \dots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_{r'} \\ &\times \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \dots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_{r'+1} \dots \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \dots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_{2sm} \\ &\times \sum_{\alpha^+(\{\varepsilon_j\})} Q(\{\varepsilon_j, \varepsilon_j'\}; \lambda_1, \dots, \lambda_{2sm}) \det S(\lambda_1, \dots, \lambda_{2sm}).\end{aligned}\tag{6.9}$$

Here the sum of $\alpha^+(\{\varepsilon_j\})$ is taken over all $\{\varepsilon_j\}$ corresponding to $\{\varepsilon_b^{[k]}\}$ ($1 \leq k \leq m$) such that the number of $\varepsilon_b^{[k]} = 1$ with $1 \leq b \leq 2s$ is given by j_k . $Q(\{\varepsilon_j, \varepsilon'_j\}; \lambda_1, \dots, \lambda_{2sm})$ is given by

$$\begin{aligned} & Q(\{\varepsilon_j, \varepsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) \\ &= (-1)^{r'} \frac{\prod_{j \in \alpha^-(\{\varepsilon'_j\})} \left(\prod_{k=1}^{j-1} \sinh(\tilde{\lambda}_j - w_k^{(2s)} + \eta) \prod_{k=j+1}^{2sm} \sinh(\tilde{\lambda}_j - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \sinh(\lambda_\ell - \lambda_k + \eta + \epsilon_{\ell,k})} \\ & \times \frac{\prod_{j \in \alpha^+(\{\varepsilon_j\})} \left(\prod_{k=1}^{j-1} \sinh(\tilde{\lambda}'_j - w_k^{(2s)} - \eta) \prod_{k=j+1}^{2sm} \sinh(\tilde{\lambda}'_j - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \sinh(w_k^{(2s)} - w_\ell^{(2s)})}. \end{aligned} \quad (6.10)$$

In the denominator we set $\epsilon_{k,\ell} = i\epsilon$ for $Im(\lambda_k - \lambda_\ell) > 0$ and $\epsilon_{k,\ell} = -i\epsilon$ for $Im(\lambda_k - \lambda_\ell) < 0$, where ϵ is an infinitesimally small positive number. The coefficient $\hat{C}^{(2s)}(\{i_k, j_k\})$ is given by

$$\begin{aligned} \hat{C}^{(2s)}(\{i_k, j_k\}) &= \prod_{k=1}^m \hat{N}_{i_k, j_k}^{(\ell)} \\ &= \prod_{k=1}^m \left(\frac{g(j_k)}{g(i_k)} \frac{F(2s, i_k)}{F(2s, j_k)} q^{i_k(2s-i_k)/2 - j_k(2s-j_k)/2} \right). \end{aligned} \quad (6.11)$$

Here we have made use of (4.30) and (5.19). If we put $g(2s, j) = \sqrt{F(2s, j)}$ for $j = 0, 1, \dots, 2s$ into (6.11), we have

$$\hat{C}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^m \sqrt{\left[\begin{matrix} 2s \\ i_k \end{matrix} \right]_q \left[\begin{matrix} 2s \\ j_k \end{matrix} \right]_q^{-1}}. \quad (6.12)$$

In (6.10) we may take any $\alpha^-(\{\varepsilon'_j\})$ corresponding to $\varepsilon_\alpha^{[k]}'$ s for $k = 1, 2, \dots, m$, as far as the number of $\varepsilon_\alpha^{[k]}' = 1$ with $1 \leq \alpha \leq 2s$ is given by i_k for each k .

We can show the symmetric expression for the multiple-integral representation of the spin- s correlation function $\hat{F}_m^{(2s, w)}(\{i_k, j_k\})$ as follows.

$$\begin{aligned} \hat{F}_m^{(2s, w)}(\{i_k, j_k\}) &= \frac{\hat{C}^{(2s)}(\{i_k, j_k\})}{\prod_{1 \leq \alpha < \beta \leq 2s} \sinh^m(\beta - \alpha)\eta} \prod_{1 \leq k < \ell \leq m} \frac{\sinh^{2s}(\pi(\xi_k - \xi_\ell)/\zeta)}{\prod_{j=1}^{2s} \prod_{r=1}^{2s} \sinh(\xi_k - \xi_\ell + (r-j)\eta)} \\ & \times \sum_{\sigma \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}} (\text{sgn } \sigma) \prod_{j=1}^{r'} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\mu_{\sigma j} \prod_{j=r'+1}^{2sm} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\mu_{\sigma j} \\ & \sum_{\{\varepsilon_b^{[1]}\}} \cdots \sum_{\{\varepsilon_b^{[k]}\}} Q'(\{\varepsilon_j, \varepsilon'_j\}; \lambda_{\sigma 1}, \dots, \lambda_{\sigma(2sm)}) \left(\prod_{j=1}^{2sm} \frac{\prod_{b=1}^m \prod_{\beta=1}^{2s-1} \sinh(\lambda_j - \xi_b + \beta\eta)}{\prod_{b=1}^m \cosh(\pi(\mu_j - \xi_b)/\zeta)} \right) \\ & \times \frac{i^{2sm^2}}{(2i\zeta)^{2sm}} \prod_{\gamma=1}^{2s} \prod_{1 \leq b < a \leq m} \sinh(\pi(\mu_{2s(a-1)+\gamma} - \mu_{2s(b-1)+\gamma})/\zeta). \end{aligned} \quad (6.13)$$

Here λ_j are given by $\lambda_j = \mu_j - (\beta(j) - 1/2)\eta$ for $j = 1, \dots, 2sm$, and $(\text{sgn } \sigma)$ denotes the sign of permutation $\sigma \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$. The coefficient $\hat{C}^{(2s)}(\{i_k, j_k\})$ is given by (6.11), and $Q'(\{\varepsilon_j, \varepsilon'_j\}; \lambda_1, \dots, \lambda_{2sm})$ is given by $Q'(\{\varepsilon_j, \varepsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) = Q(\{\varepsilon_j, \varepsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) \times$

$\prod_{1 \leq k < \ell \leq 2sm} \sinh(w_k^{(2s)} - w_\ell^{(2s)})$. We recall that the sums over $\{\varepsilon_\beta^{(k)}\}$ are taken over all $\varepsilon_\beta^{(k)}$ s for $1 \leq k \leq m$ such that the number of integers β with $\varepsilon_\beta^{(k)} = 1$ and $1 \leq \beta \leq \ell$ is equal to j_k for each k .

In Appendix E we shall explain the derivation of the symmetric expression for the multiple-integral representation of the correlation functions.

6.3 Relations due to the spin inversion symmetry

We shall derive a consequence of the spin-inversion symmetry. We shall assume that it should hold for the ground state of the integrable spin- s XXZ chain associated with even L . Here we recall $L = 2sN_s$ and N_s is the number of the lattice sites of the spin- $\ell/2$ XXZ chain and L that of the spin-1/2 XXZ chain associated with it.

Let us denote by $|\psi_g^{(2s w; 0)}\rangle$ the Bethe-ansatz eigenstate with $M = sN_s$ down-spins for the spin-1/2 XXZ transfer matrix under zero magnetic field. It is given by $|\psi_g^{(2s w; 0)}\rangle = \prod_{\gamma=1}^M B^{(2s w; 0)}(\lambda_\gamma)|0\rangle$ with $M = L/2$ where inhomogeneity parameters are given by the N_s pieces of the complete $2s$ -strings, $w_j^{(2s)}$ for $j = 1, 2, \dots, L$. We now assume that it has the spin inversion symmetry:

$$U|\psi_g^{(2s w; 0)}\rangle = \pm |\psi_g^{(2s w; 0)}\rangle \quad \text{for} \quad U = \prod_{j=1}^L \sigma_j^x. \quad (6.14)$$

It leads to symmetry relations as follows.

$$\langle \psi_g^{(2s w; 0)} | e_1^{\varepsilon'_1, \varepsilon_1} \dots e_{2s}^{\varepsilon'_{2s}, \varepsilon_{2s}} | \psi_g^{(2s w; 0)} \rangle = \langle \psi_g^{(2s w; 0)} | e_1^{1-\varepsilon'_1, 1-\varepsilon_1} \dots e_{2s}^{1-\varepsilon'_{2s}, 1-\varepsilon_{2s}} | \psi_g^{(2s w; 0)} \rangle. \quad (6.15)$$

For the XXX case where the parameter q is given by 1, we can show that the spin-1/2 XXX transfer matrix with arbitrary inhomogeneity parameters w_j has the $SU(2)$ symmetry and hence every Bethe-ansatz eigenvector is a highest weight vector of the $SU(2)$. Thus, the Bethe eigenvector with $S^Z = 0$ has the total spin 0, and hence it is invariant under any rotational operation. Therefore, it has the spin inversion symmetry. For the XXZ case where q is not equal to 1, the spin-1/2 XXX transfer matrix with arbitrary inhomogeneity parameters w_j does not have the $SU(2)$ symmetry, in general. However, we assume that it has the spin inversion symmetry such as the XXX case.

Applying the spin-inversion symmetry (6.15) we derive symmetry relations among the expectation values of local or global operators.

For an illustration, let us evaluate the one-point function in the spin-1 case with $i_1 = j_1 = 1$, $\langle E_1^{1, 1(2p)} \rangle$. Setting $\varepsilon'_1 = 0$ and $\varepsilon'_2 = 1$ we decompose the spin-1 elementary operator in terms of a sum of products of the spin-1/2 ones

$$\langle \psi_g^{(2p)} | E_1^{1, 1(2p)} | \psi_g^{(2p)} \rangle = \langle \psi_g^{(2p; 0)} | e_1^{0, 0} e_2^{1, 1} | \psi_g^{(2p; 0)} \rangle + \langle \psi_g^{(2p; 0)} | e_1^{0, 1} e_2^{1, 0} | \psi_g^{(2p; 0)} \rangle. \quad (6.16)$$

Through the symmetry relations (4.13) with respect to ε'_α we have the following equalities:

$$\begin{aligned}\langle \psi_g^{(2p;0)} | e_1^{0,0} e_2^{1,1} | \psi_g^{(2p;0)} \rangle &= \langle \psi_g^{(2p;0)} | e_1^{1,0} e_2^{0,1} | \psi_g^{(2p;0)} \rangle, \\ \langle \psi_g^{(2p;0)} | e_1^{1,1} e_2^{0,0} | \psi_g^{(2p;0)} \rangle &= \langle \psi_g^{(2p;0)} | e_1^{0,1} e_2^{1,0} | \psi_g^{(2p;0)} \rangle.\end{aligned}\quad (6.17)$$

From spin-inversion symmetry (6.15) we have

$$\begin{aligned}\langle \psi_g^{(2p;0)} | e_1^{0,0} e_2^{1,1} | \psi_g^{(2p;0)} \rangle &= \langle \psi_g^{(2p;0)} | e_1^{1,1} e_2^{0,0} | \psi_g^{(2p;0)} \rangle, \\ \langle \psi_g^{(2p;0)} | e_1^{0,1} e_2^{1,0} | \psi_g^{(2p;0)} \rangle &= \langle \psi_g^{(2p;0)} | e_1^{1,0} e_2^{0,1} | \psi_g^{(2p;0)} \rangle\end{aligned}\quad (6.18)$$

and hence we have the equalities of the four terms. We therefore obtain the following:

$$\langle \psi_g^{(2)} | E_1^{1,1(2p)} | \psi_g^{(2)} \rangle = 2 \langle \psi_g^{(2p;0)} | e_1^{0,0} e_2^{1,1} | \psi_g^{(2p;0)} \rangle. \quad (6.19)$$

We thus derive the double-integral representation of the one-point function $\langle E_1^{1,1(2p)} \rangle$ given in Ref. [13]. For the spin-1 case, each of the one-point functions is given by the double integral associated with a single product of the spin-1/2 operators.

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A Derivation of Proposition 3.7

It follows from Lemma 3.1 that we have $E^{i,j(\ell w)} = P^{(\ell)} E^{i,j(\ell w)}$. We therefore have

$$\begin{aligned}E^{i,j(\ell w)} &= P^{(\ell)} \sum_{(\varepsilon'_\alpha(i))_\ell} \sum_{(\varepsilon_\beta(j))_\ell} g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- ||\ell, 0\rangle \langle \ell, 0| \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ \\ &= ||\ell, i\rangle \sum_{(\varepsilon'_\alpha(i))_\ell} \sum_{(\varepsilon_\beta(j))_\ell} g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) \langle \ell, i| \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- ||\ell, 0\rangle \langle \ell, 0| \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ \\ &= ||\ell, i\rangle \sum_{(\varepsilon_\beta(i))_\ell} \sum_{(\varepsilon'_\alpha(j))_\ell} g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) q^{a(1)+\cdots+a(i)-i} \\ &\quad \times \langle \ell, i| \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- ||\ell, 0\rangle q^{-(a(1)+\cdots+a(i)-i)} \langle \ell, 0| \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+\end{aligned}\quad (A.1)$$

Applying Lemma 3.3 we have

$$\begin{aligned}E^{i,j(\ell w)} &= ||\ell, i\rangle \sum_{(\varepsilon_\beta(i))_\ell} \left(\sum_{(\varepsilon'_\alpha(j))_\ell} g_{ij}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) q^{a(1)+\cdots+a(i)-i} \right) \\ &\quad \times \langle \ell, i| \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- ||\ell, 0\rangle q^{-(a(1)+\cdots+a(i)-i)} \langle \ell, 0| \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+\end{aligned}\quad (A.2)$$

Taking the sum over sequences $(\varepsilon'_\alpha(i))_\ell$ with Lemma 3.6 we have

$$\begin{aligned}
E^{i,j(\ell w)} &= ||\ell, i\rangle \begin{bmatrix} \ell \\ i \end{bmatrix} \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} q^{i(i-1)/2-j(j-1)/2} \langle \ell, i || \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- || \ell, 0 \rangle q^{-(a(1)+\cdots+a(i)-i)} \\
&\quad \times \sum_{(\varepsilon_\beta(j))_\ell} \langle \ell, 0 || \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ q^{b(1)+\cdots+b(j)-j}
\end{aligned} \tag{A.3}$$

We move the conjugate vector $\langle \ell, i ||$ to the left in (A.3), and through (3.15) we express $\sigma_{a(1)}^- \cdots \sigma_{a(i)}^- || \ell, 0 \rangle \langle \ell, 0 || \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+$ in terms of the spin-1/2 elementary operators, we have

$$\begin{aligned}
E^{i,j(\ell w)} &= ||\ell, i\rangle \langle \ell, i || \begin{bmatrix} \ell \\ i \end{bmatrix} \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} q^{i(i-1)/2-j(j-1)/2} \\
&\quad \times \sum_{(\varepsilon_\beta(j))_\ell} e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \cdots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)} q^{-(a(1)+\cdots+a(i)-i)} q^{b(1)+\cdots+b(j)-j}
\end{aligned} \tag{A.4}$$

Applying the gauge transformation of Lemma 3.4 to (A.4) we obtain Proposition 3.7.

B Reduction of spin- $\ell/2$ Hermitian elementary operators

Let us introduce vectors $\widetilde{||\ell, n\rangle}$ which are Hermitian conjugate to $\langle \ell, n ||$ when $|q| = 1$ for positive integers ℓ with $n = 0, 1, \dots, \ell$ [13]. Setting the norm of $\widetilde{||\ell, n\rangle}$ such that $\langle \ell, n || \widetilde{||\ell, n\rangle} = 1$, vectors $\widetilde{||\ell, n\rangle}$ are given by

$$\widetilde{||\ell, n\rangle} = \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- ||\ell, 0\rangle q^{-(i_1+\cdots+i_n)+n\ell-n(n-1)/2} \begin{bmatrix} \ell \\ n \end{bmatrix}_q q^{-n(\ell-n)} \begin{pmatrix} \ell \\ n \end{pmatrix}^{-1}. \tag{B.1}$$

We define the spin- $\ell/2$ Hermitian elementary matrices associated with homogeneous grading, $\widetilde{E}^{i,j(\ell,+)}$, by

$$\widetilde{E}^{i,j(\ell,+)} = \widetilde{||\ell, i\rangle} \langle \ell, j ||. \tag{B.2}$$

Introducing $\widetilde{g}_{i,j}$ by

$$\widetilde{||\ell, i\rangle} \langle \ell, j || = \sum_{(\varepsilon'_\alpha(i))_\ell} \sum_{(\varepsilon_\beta(j))_\ell} \widetilde{g}_{i,j}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) e_1^{\varepsilon'_1(i), \varepsilon_1(j)} \cdots e_\ell^{\varepsilon'_\ell(i), \varepsilon_\ell(j)}, \tag{B.3}$$

we have

$$\widetilde{g}_{i,j}(\varepsilon'_\alpha(i), \varepsilon_\beta(j)) = \begin{bmatrix} \ell \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} \begin{pmatrix} \ell \\ i \end{pmatrix}_q q^{i(i-1)/2-j(j-1)/2} q^{-(a(1)+\cdots+a(i)-i)+(b(1)+\cdots+b(j)-j)} \tag{B.4}$$

We derive the reduction formula for the Hermitian elementary operators $\widetilde{E}^{i,j(\ell,+)}$ as follows.

$$\begin{aligned}
\widetilde{E}^{i,j(\ell,+)} &= \widetilde{P}^{(\ell)} \widetilde{E}^{i,j(\ell,+)} \\
&= \begin{bmatrix} \ell \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} \begin{pmatrix} \ell \\ i \end{pmatrix} q^{i(i-1)/2-j(j-1)/2} \widetilde{||\ell, i\rangle} \\
&\times \sum_{(\varepsilon_\beta(j))_\ell} \sum_{(\varepsilon'_\alpha(i))_\ell} \left(\langle \ell, i | \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- | \ell, 0 \rangle q^{-(a(1)+\cdots+a(i)-i)} \right) \langle \ell, 0 | \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ q^{b(1)+\cdots+b(j)-i} .
\end{aligned} \tag{B.5}$$

Here, applying Lemma 3.3 we show that the inside of the parentheses (or the round brackets) is independent of $a(k)$ s. Making use of the following:

$$\sum_{(\varepsilon'_\alpha(i))_\ell} 1 = \begin{pmatrix} \ell \\ i \end{pmatrix} , \tag{B.6}$$

we thus have

$$\begin{aligned}
\widetilde{E}^{i,j(\ell,+)} &= \begin{bmatrix} \ell \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} \begin{pmatrix} \ell \\ i \end{pmatrix}^{-1} q^{i(i-1)/2-j(j-1)/2} \widetilde{||\ell, i\rangle} \langle \ell, i | \\
&\times \begin{pmatrix} \ell \\ i \end{pmatrix} \sum_{(\varepsilon_\beta(j))_\ell} \sigma_{a(1)}^- \cdots \sigma_{a(i)}^- | \ell, 0 \rangle \langle \ell, 0 | \sigma_{b(1)}^+ \cdots \sigma_{b(j)}^+ q^{-(a(1)+\cdots+a(i)-i)} q^{b(1)+\cdots+b(j)-i} \\
&= \begin{bmatrix} \ell \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} q^{i(i-1)/2-j(j-1)/2} \widetilde{||\ell, i\rangle} \langle \ell, i | e^{-(i-j)\xi_1} \sum_{\{\varepsilon_\beta\}} \chi_{1\dots\ell} e_1^{\varepsilon'_1, \varepsilon_1} \cdots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell} \chi_{1\dots\ell}^{-1} .
\end{aligned} \tag{B.7}$$

Here we have applied Lemma 3.4 to derive the last line of eq. (B.7).

C Non-regularity of the transfer matrix

Let us consider the case of $L = 3$. We introduce b_{0j} and c_{0j}^\pm for $j = 1, 2, 3$ by $b_{0j} = b(\lambda - w_j^{(2)})$ and $c_{0j}^\pm = \exp(\pm(\lambda - w_j^{(2)}))c(\lambda - w_j^{(2)})$ for $j = 1, 2, 3$, respectively.

The matrix elements of the operator $A_{123}^{(1+)}(\lambda)$ in the sector of $M = 1$ are given by

$$A_{123}^{(1+)}(\lambda) \Big|_{M=1} = \begin{pmatrix} b_{03} & c_{02}^+ c_{03}^- & c_{01}^+ b_{02} c_{03}^- \\ 0 & b_{02} & c_{01}^+ c_{02}^- \\ 0 & 0 & b_{01} \end{pmatrix} , \tag{C.1}$$

and those of the operator $A_{123}^{(1+)}(\lambda)$ in the sector of $M = 1$ are given by

$$D_{123}^{(3+;0)}(\lambda) \Big|_{M=1} = \begin{pmatrix} b_{01} b_{02} & 0 & \\ b_{01} c_{02}^- c_{03}^+ & b_{01} b_{03} & 0 \\ c_{01}^- c_{03}^+ & c_{01}^+ c_{02}^+ b_{03} & b_{02} b_{03} \end{pmatrix} . \tag{C.2}$$

Let us set $w_1 = w_1^{(2)} = \xi_1$, $w_2 = w_2^{(2)} = \xi_1 - \eta$, and $w_3 = \xi_2$. Setting $\lambda = \xi_1$ we have

$$\left(A_{123}^{(2+;0)}(\xi_1) + D_{123}^{(2+;0)}(\xi_1) \right) \Big|_{S^Z=1/2} = \begin{pmatrix} b_{13} & \frac{q}{[2]_q} c_{13}^- & \frac{1}{[2]_q} c_{13}^- \\ 0 & \frac{1}{[2]_q} & \frac{q^{-1}}{[2]_q} \\ c_{13}^+ & \frac{q}{[2]_q} b_{13} & \frac{1}{[2]_q} b_{13} \end{pmatrix}. \quad (\text{C.3})$$

Here, the second and the third columns are parallel. Thus, the determinant of the spin-1/2 transfer matrix in the sector of $M = 1$ is non-regular.

D Reducing spin- $\ell/2$ Bethe states with principal grading

In order to evaluate the spin- s form factors for the spin- s elementary operators $E^{i,j(\ell p)}$ associated with principal grading, we first transform them into those of homogeneous grading, and then apply to them the formula for expressing the spin- s elementary operators in terms of a sum of products of spin-1/2 elementary operators.

Let us recall the gauge transformation $\chi_{0,12\dots N_s}^{(1,\ell)}$ which maps the higher-spin transfer matrix associated with principal grading of type $(1, \ell^{\otimes N_s})$ to that of homogeneous grading:

$$\begin{aligned} T_{0,12\dots N_s}^{(1,\ell+)}(\lambda) &= \chi_{0,12\dots N_s}^{(1,\ell)} T^{(1,\ell p)}(\lambda) \left(\chi_{0,12\dots N_s}^{(1,\ell)} \right)^{-1} \\ &= \begin{pmatrix} \chi_{12\dots N_s}^{(\ell)} A^{(\ell p)}(\lambda) \chi_{12\dots N_s}^{(\ell)-1} & e^{-\lambda} \chi_{12\dots N_s}^{(\ell)} B^{(\ell p)}(\lambda) \chi_{12\dots N_s}^{(\ell)-1} \\ e^{\lambda} \chi_{12\dots N_s}^{(\ell)} C^{(\ell p)}(\lambda) \chi_{12\dots N_s}^{(\ell)-1} & \chi_{12\dots N_s}^{(\ell)} D^{(\ell p)}(\lambda) \chi_{12\dots N_s}^{(\ell)-1} \end{pmatrix}. \end{aligned} \quad (\text{D.1})$$

We also recall that the C operator acting on the tensor product of the spin- $\ell/2$ representations $(V^{(\ell)})^{\otimes N_s}$ is derived from the C operator acting on the tensor product of the spin-1/2 representations $(V^{(1)})^{\otimes L}$ multiplied by the projection operators:

$$C^{(\ell+)}(\mu) = P_{12\dots L}^{(\ell)} C^{(\ell+;0)}(\mu) P_{12\dots L}^{(\ell)} e^{\mu}.$$

Here, through the spin-1/2 gauge transformation we have

$$C^{(\ell+;0)}(\mu) = \chi_{12\dots L} C^{(\ell p;0)}(\mu) \chi_{12\dots L}^{-1} e^{\mu}.$$

We therefore have $\langle \{\mu_\alpha\}_N^{(\ell,p)} |$ as

$$\begin{aligned} \langle 0 | \prod_{\alpha=1}^N C^{(\ell p)}(\mu_\alpha) &= \langle 0 | \prod_{\alpha=1}^N \left(\left(\chi_{1\dots N_s}^{(\ell)} \right)^{-1} P_{1\dots L}^{(\ell)} \chi_{1\dots L} C^{(\ell p;0)}(\mu_\alpha) \chi_{1\dots L}^{-1} P_{1\dots L}^{(\ell)} \left(\chi_{1\dots N_s}^{(\ell)} \right) \right) \\ &= \langle 0 | \prod_{k=1}^N C^{(\ell p;0)}(\mu_k) \cdot \chi_{1\dots L}^{-1} P_{1\dots L}^{(\ell)} \chi_{1\dots N_s}^{(\ell)}. \end{aligned} \quad (\text{D.2})$$

Precisely we derive it as follows:

$$\begin{aligned}
\langle 0 | \prod_{k=1}^N C^{(\ell p)}(\mu_k) &= \langle 0 | \prod_{k=1}^N \left\{ \left(\chi_{1 \dots N_s}^{(\ell)} \right)^{-1} C^{(\ell+)}(\mu_k) e^{-\mu_k} \left(\chi_{1 \dots N_s}^{(\ell)} \right) \right\} \\
&= \langle 0 | \prod_{k=1}^N \{ C^{(\ell+)}(\mu_k) e^{-\mu_k} \} \cdot \chi_{1 \dots N_s}^{(\ell)} \\
&= \langle 0 | \prod_{k=1}^N \left(P_{1 \dots L}^{(\ell)} C^{(\ell+;0)}(\mu_k) e^{-\mu_k} P_{1 \dots L}^{(\ell)} \right) \cdot \chi_{1 \dots N_s}^{(\ell)} \\
&= \langle 0 | \prod_{k=1}^N \{ C^{(\ell+;0)}(\mu_k) e^{-\mu_k} \} \cdot P_{1 \dots L}^{(\ell)} \chi_{1 \dots N_s}^{(\ell)}. \tag{D.3}
\end{aligned}$$

Here we have made use of the commutation relation with the projection $P_{1 \dots L}^{(\ell)}$. Thus, we have

$$\begin{aligned}
|\{\mu_\alpha\}_N^{(\ell, p)}| &= \langle 0 | \prod_{k=1}^N (\chi_{1 \dots L} C^{(\ell p;0)}(\mu_k) \chi_{1 \dots L}^{-1}) \cdot P_{1 \dots L}^{(\ell)} \chi_{1 \dots N_s}^{(\ell)} \\
&= \langle 0 | \prod_{k=1}^N C^{(\ell p;0)}(\mu_k) \cdot \chi_{1 \dots L}^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots N_s}^{(\ell)}. \tag{D.4}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
|\{\mu_\alpha\}_N^{(\ell, p)}\rangle &= \prod_{\alpha=1}^N B^{(\ell p)}(\lambda_\alpha) |0\rangle \\
&= \prod_{\alpha=1}^N \left(\left(\chi_{1 \dots N_s}^{(\ell)} \right)^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots L} B^{(\ell p;0)}(\lambda_\alpha) \chi_{1 \dots L}^{-1} P_{1 \dots L}^{(\ell)} \left(\chi_{1 \dots N_s}^{(\ell)} \right) \right) \\
&= \left(\chi_{1 \dots N_s}^{(\ell)} \right)^{-1} P_{1 \dots L}^{(\ell)} \chi_{1 \dots L} \cdot \prod_{\alpha=1}^M B^{(\ell p;0)}(\lambda_\alpha) |0\rangle. \tag{D.5}
\end{aligned}$$

E Symmetric multiple-integral representations

For the spin-1 case, let us express the double sum $\sum_{c_1}^{M'} \sum_{c_2}^M f(c_1, c_2)$ in the symmetric form which leads to the symmetric expression of the multiple-integral representations. Here c_1 and c_2 run through from 1 to M corresponding to all the 2-strings of the ground state.

Recall that variable c_j ($1 \leq j \leq 2sm$) takes integers from 1 to $M = N_s$ which correspond to $N_s/2$ sets of 2-strings. We express them in terms of integers $a(j, \beta)$ for $\beta = 1, 2$, where $a(j, \beta)$ take integral values from 1 to $N_s/2$. We first express the sum over c_1 in terms of $a(1, \beta)$ as follows.

$$\sum_{c_1} = \sum_{a(1,1)=1}^{M/2} + \sum_{a(1,2)=1}^{M/2} \tag{E.1}$$

More precisely we have

$$\sum_{c_1} f(c_1) = \sum_{a(1,1)=1}^{M/2} f(2(a(1,1) - 1) + 1) + \sum_{a(1,2)=1}^{M/2} f(2(a(1,2) - 1) + 1) \quad (\text{E.2})$$

For the spin-1 case with one-point function ($m = 1$) we have

$$\begin{aligned} \sum_{c_1=1}^{M'} \sum_{c_2=1}^M f(c_1, c_2) &= \left(\sum_{a(1,1)=1}^{M'/2} + \sum_{a(1,2)=1}^{M'/2} \right) \left(\sum_{a(2,1)=1}^{M/2} + \sum_{a(2,2)=1}^{M/2} \right) f(c_1, c_2) \\ &= \left(\sum_{a(1,1)=1}^{M'/2} \sum_{a(2,1)=1}^{M/2} + \sum_{a(1,1)=1}^{M'/2} \sum_{a(2,2)=1}^{M/2} + \sum_{a(1,2)=1}^{M'/2} \sum_{a(2,1)=1}^{M/2} + \sum_{a(1,2)=1}^{M'/2} \sum_{a(2,2)=1}^{M/2} \right) f(c_1, c_2) \\ &= \left(0 + \sum_{a(1,1)=1}^{M'/2} \sum_{a(2,2)=1}^{M/2} + \sum_{a(1,2)=1}^{M'/2} \sum_{a(2,1)=1}^{M/2} + 0 \right) f(c_1, c_2) \end{aligned} \quad (\text{E.3})$$

Here we recall that $f(c_1, c_2)$ vanishes if the types of string rapidities c_1 and c_2 are the same.

Let us now introduce variables a_j ($j = 1, 2$) which correspond to the centers of the 2-strings. We define an integer-valued variable \hat{c}_j which is a function of a_j as follows

$$\hat{c}_j = 2(a_j - 1) + \beta(j) \quad (\text{E.4})$$

Then, in terms of permutations π in the symmetric group \mathcal{S}_2 we express the sum as follows

$$\begin{aligned} \sum_{c_1}^{M'} \sum_{c_2}^M f(c_1, c_2) &= \sum_{a_1=1}^{M'/2} \sum_{a_2=1}^{M/2} f(\hat{c}_1, \hat{c}_2) + \sum_{a_2=1}^{M'/2} \sum_{a_1=1}^{M/2} f(\hat{c}_2, \hat{c}_1) \\ &= \sum_{a_{e1}=1}^{M'/2} \sum_{a_{e2}=1}^{M/2} f(\hat{c}_{e1}, \hat{c}_{e2}) + \sum_{a_{(12)1}=1}^{M'/2} \sum_{a_{(12)2}=1}^{M/2} f(\hat{c}_{(12)1}, \hat{c}_{(12)2}) \\ &= \sum_{\pi \in \mathcal{S}} \sum_{a_{\pi 1}=1}^{M'/2} \sum_{a_{\pi 2}=1}^{M/2} f(\hat{c}_{\pi 1}, \hat{c}_{\pi 2}). \end{aligned} \quad (\text{E.5})$$

We thus have

$$\sum_{c_1=1}^{M'} \sum_{c_2=1}^M f(c_1, c_2) = \sum_{\pi \in \mathcal{S}} \sum_{a_{\pi 1}=1}^{M'/2} \sum_{a_{\pi 2}=1}^{M/2} f(\hat{c}_{\pi 1}, \hat{c}_{\pi 2}) \quad (\text{E.6})$$

The result leads to the symmetric expression of the multiple-integral representation.

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- [19] Here we assume that for a given solution $\{\lambda_\gamma\}$ of the the Bethe ansatz equations of the spin- $\ell/2$ chain there is a solution of the Bethe ansatz equations, $\{\lambda_\gamma(\epsilon)\}$, for the spin-1/2 chain with inhomogeneity parameters given by the almost complete ℓ -strings, i.e., $w_j = w_j^{(\ell; \epsilon)}$ ($1 \leq j \leq L$), and also that the solution $\{\lambda_\gamma(\epsilon)\}$ approaches $\{\lambda_\gamma\}$ continuously at $\epsilon = 0$.
- [20] In §4.3 of Ref. [12] the spin- $\ell/2$ form factors were discussed by making use of the invalid QISP formulas, and the following expressions are not valid: [4.40] and [4.42] for $X^{-(\ell+)}$; [4.44] and [4.48] for $K^{(\ell)}$; [4.50] for S_i^Z ; [4.52] for $E^{m, m^{(\ell)}}$ and [4.54] for $E^{m-p-1, m^{(\ell)}}$. Here [a.bc] denote eq. (a.bc) of Ref. [12]; See also, T. Deguchi and C. Matsui, Erratum to “Form factors of integrable higher-spin XXZ chains and the affine quantum-group symmetry”, *Nucl. Phys. B* 814 (2009) 405–438, to appear in *Nuclear Physics B*.
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